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A note on confidence interval estimation for a linear function of binomial proportions

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ABSTRACT

The Wilson score confidence interval for a binomial proportion has been widely applied in practice, due largely to its good performance in finite samples and its simplicity in calculation. We propose its use in setting confidence limits for a linear function of binomial proportions using the method of variance estimates recovery. Exact evaluation results show that this approach provides intervals that are narrower than the ones based on the adjusted Wald interval while aligning the mean coverage with the nominal level.

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1. Introduction

There exists a large literature on confidence interval estimation involving binomial proportions. For a single proportion, there are several choices. The first is given by adding to and subtracting from the maximum likelihood estimator the standard normal quantile multiplied by its estimated standard error. This procedure is commonly referred to as the Wald method. The second is the interval based on inverting the approximate normal test that uses the standard errors estimated at the lower and upper limits. This procedure is commonly referred to as Wilson score method (Wilson, 1927). The score confidence interval has now become very popular, especially after the expositions by Agresti and Coull (1998) and Newcombe (1998b). With an attempt to ease classroom teaching, Agresti and Coull (1998) suggested an adjusted Wald method by adding two successes and two failures and then using the Wald formula. Despite the terminology, the adjusted Wald method is actually an approximation of the score method.

The superior performance of the Wilson method has been carried over to cases of a difference between two proportions (Newcombe, 1998a) and a difference between two differences (Newcombe, 2001). It is interesting to note that this seemingly *ad hoc* procedure has become more popular than the rigorous score interval for a difference between two proportions (Mee, 1984; Miettinen and Nurminen, 1985; Gart and Nam, 1990), caused largely by the computation involved in obtaining the latter.

Due to its important practical value, confidence interval construction for a linear function of binomial proportions has received some attention recently (Price and Bonett, 2004; Tebbs and Roths, 2008). The purpose of this note is to extend the argument of Zou and Donner (2008) to a linear function of parameters, and in particular to binomial proportions. Since our main idea is to recover variance estimates from readily available confidence limits for single parameters, we refer to the approach as the MOVER, the method of variance estimates recovery. As shown below, the MOVER will not only shed some light to Newcombe (1998a) and Newcombe (2001) but also provide an alternative to Price and Bonett (2004) who proposed

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a procedure for a linear function of proportions based on the adjusted Wald interval for single proportions (Agresti and Coull, 1998). We will show that the confidence interval for a linear function of binomial proportions based on the Wilson method is narrower than that of Price and Bonett (2004). We will not consider the approach by Tebbs and Roths (2008) because of its inherent drawbacks such as involved computation, restriction in parameter ranges, and undercoverage.

2. The MOVER and its application to linear functions of binomial proportions

Suppose we wish to construct an approximate $100(1-\alpha)\%$ two-sided confidence interval for $\theta_1+\theta_2$, where the estimates $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are assumed to be independent. By the central limit theorem, the lower limit (L) is given by

$$L = \widehat{\theta}_1 + \widehat{\theta}_2 - z_{\alpha/2}\sqrt{\mathrm{var}(\widehat{\theta}_1) + \mathrm{var}(\widehat{\theta}_2)}. \tag{1}$$

Inspired by the score method for interval estimation (Bartlett, 1953; Gart and Nam, 1990), we can estimate the variance needed for L at $\theta_1 + \theta_2 = L$. This has at least one disadvantage that it is in general an iterative procedure, which can be an obstacle to wide application in practice as what happened to the score interval for a difference between two proportions. Therefore, we proceed with estimating the variance in the neighborhood of L.

Now, suppose that the $100(1-\alpha)\%$ two-sided confidence intervals (l_i, u_i) for single parameters θ_i, $i = 1, 2$ are available. Note that there is no need to specify the approaches taken to obtain (l_i, u_i). Among all the plausible parameter values of θ_1 provided by (l_1, u_1) and that of θ_2 provided by (l_2, u_2), $l_1 + l_2$ is usually closer to L than $\widehat{\theta}_1 + \widehat{\theta}_2$. As a result, for L, we can estimate $\mathrm{var}(\widehat{\theta}_1)$ at $\theta_1 = l_1$ and $\mathrm{var}(\widehat{\theta}_2)$ at $\theta_2 = l_2$.

Furthermore, we can recover the required variance estimates from $\hat{\theta}_i(l_i, u_i)$, $i = 1, 2$, as follows. By the central limit theorem and letting $z_{\alpha/2}$ be the upper $\alpha/2$ quantile of the standard Normal distribution, we have

$$l_i = \widehat{\theta}_i - z_{\alpha/2}\sqrt{\widehat{\mathrm{var}}(\widehat{\theta}_i)},$$

which gives a variance estimate for $\widehat{\theta}_i$ at $\theta_i = l_i$ as

$$\widehat{\mathrm{var}}_l(\widehat{\theta}_i) = (\widehat{\theta}_i - l_i)^2/z^2_{\alpha/2}$$

and

$$u_i = \widehat{\theta}_i + z_{\alpha/2}\sqrt{\widehat{\mathrm{var}}(\widehat{\theta}_i)},$$

which gives a variance estimate at $\theta_i = u_i$ as

$$\widehat{\mathrm{var}}_u(\widehat{\theta}_i) = (u_1 - \widehat{\theta}_i)^2/z^2_{\alpha/2}.$$

Note that the recovered variance estimates $\widehat{\mathrm{var}}_l(\widehat{\theta}_i)$ and $\widehat{\mathrm{var}}_u(\widehat{\theta}_i)$ are different, except when the interval (l_i, u_i) is symmetric about $\widehat{\theta}_i$. Symmetric intervals are known to perform poorly in finite samples for most problems in practice. In fact, it was stated (Efron and Tibshirani, 1993, p. 180) that symmetry is the most serious error in confidence interval construction. The Wald interval for a binomial proportion is a perfect example. In contrast, the Wilson interval is asymmetric as a consequence of estimating variances at the lower and upper limits separately.

Plugging the recovered variance estimates into Eq. (1) results in

$$\begin{aligned} L &= \widehat{\theta}_1 + \widehat{\theta}_2 - z_{\alpha/2}\sqrt{\mathrm{var}(\widehat{\theta}_1) + \mathrm{var}(\widehat{\theta}_2)} \\ &= \widehat{\theta}_1 + \widehat{\theta}_2 - z_{\alpha/2}\sqrt{(\widehat{\theta}_1 - l_1)^2/z^2_{\alpha/2} + (\widehat{\theta}_2 - l_2)^2/z^2_{\alpha/2}} \\ &= \widehat{\theta}_1 + \widehat{\theta}_2 - \sqrt{(\widehat{\theta}_1 - l_1)^2 + (\widehat{\theta}_2 - l_2)^2}. \end{aligned}$$

Analogous steps with the notion that $u_1 + u_2$ is in the vicinity of U yield the upper limit U as

$$U = \widehat{\theta}_1 + \widehat{\theta}_2 + \sqrt{(u_1 - \widehat{\theta}_1)^2 + (u_2 - \widehat{\theta}_2)^2}.$$

Rewriting $\theta_1 - \theta_2$ as $\theta_1 + (-\theta_2)$ and noting that the confidence limits for $-\theta_2$ are given by $(-u_2, -l_2)$, we obtain confidence limits for $\theta_1 - \theta_2$ as

$$L = \widehat{\theta}_1 - \widehat{\theta}_2 - \sqrt{(\widehat{\theta}_1 - l_1)^2 + (u_2 - \widehat{\theta}_2)^2}$$

and

$$U = \widehat{\theta}_1 - \widehat{\theta}_2 + \sqrt{(u_1 - \widehat{\theta}_1)^2 + (\widehat{\theta}_2 - l_2)^2}.$$

This confidence interval, apparently first presented by Howe (1974), has been applied by Newcombe (1998a) and by Newcombe (2001) to binomial proportions. There has been no analytic justification for its general applicability until recently (Zou and Donner, 2008).

Regarding $\theta_1+\theta_2$ and $\theta_1-\theta_2$ as $c_1\theta_1+c_2\theta_2$, where c_1 and c_2 are constants, we can rewrite the intervals as

$$L=c_1\widehat{\theta}_1+c_2\widehat{\theta}_2-\sqrt{[c_1\widehat{\theta}_1-\min(c_1l_1,c_1u_1)]^2+[c_2\widehat{\theta}_2-\min(c_2l_2,c_2u_2)]^2}$$

and

$$U=c_1\widehat{\theta}_1+c_2\widehat{\theta}_2+\sqrt{[c_1\widehat{\theta}_1-\max(c_1l_1,c_1u_1)]^2+[c_2\widehat{\theta}_2-\max(c_2l_2,c_2u_2)]^2}.$$

For a $100(1-\alpha)\%$ confidence interval for $\sum_{i=1}^{g}c_i\theta_i$, where $g>2$, an application of mathematical induction results in

$$\begin{cases} L=\displaystyle\sum_{i=1}^{g}c_i\widehat{\theta}_i-\sqrt{\sum_{i=1}^{g}\left[c_i\widehat{\theta}_i-\min(c_il_i,c_iu_i)\right]^2} \\ U=\displaystyle\sum_{i=1}^{g}c_i\widehat{\theta}_i+\sqrt{\sum_{i=1}^{g}\left[c_i\widehat{\theta}_i-\max(c_il_i,c_iu_i)\right]^2}. \end{cases} \tag{2}$$

Because L and U are derived using the recovered variance estimates, we can refer to the method as the MOVER, standing for method of variance estimates recovery. A further extension of the MOVER to incorporate dependence between $\widehat{\theta}_i$ and $\widehat{\theta}_j$ $(i\neq j)$ has been applied to measures of additive interaction in epidemiology (Zou, 2008).

We can now apply the confidence interval in (2) to linear functions of binomial proportions. Since there are at least three intervals for a single proportion, i.e., Wald, adjusted Wald (Agresti and Coull, 1998) and Wilson, we end up with three procedures for linear functions of binomial proportions.

Specifically, let Y_i $(i=1,2,\ldots,g)$ be independent binomial variates with parameters (n_i,p_i), and let $\widehat{p}_i=Y_i/n_i$ be the sample estimates for p_i. A linear function of binomial proportions may be defined as $\sum_{i=1}^{g}c_ip_i$, where the c_i are known constants. Using the equations in (2), the $100(1-\alpha)\%$ Wald confidence interval can be obtained by setting $\widehat{\theta}_i=\widehat{p}_i=Y_i/n_i$, $l_i=\widehat{p}_i-z_{\alpha/2}\sqrt{\widehat{p}_i(1-\widehat{p}_i)/n_i}$, and $u_i=\widehat{p}_i+z_{\alpha/2}\sqrt{\widehat{p}_i(1-\widehat{p}_i)/n_i}$.

The Wilson interval for $\sum_{i=1}^{g}c_ip_i$ may be obtained by setting $\widehat{\theta}_i=\widehat{p}_i=Y_i/n_i$,

$$l_i,u_i=\left(\widehat{p}_i+z^2_{\alpha/2}/(2n_i)\mp z_{\alpha/2}\sqrt{[\widehat{p}_i(1-\widehat{p}_i)+z^2_{\alpha/2}/(4n_i)]/n_i}\right)/(1+z^2_{\alpha/2}/n_i).$$

The adjusted Wald interval for $\sum_{i=1}^{g}c_ip_i$ (Price and Bonett, 2004) may be obtained by setting $\widehat{\theta}_i=\tilde{p}_i=(Y_i+2/k)/(n_i+4/k)$ (where k is the number of nonzero elements in c_i), $l_i=\tilde{p}_i-z_{\alpha/2}\sqrt{\tilde{p}_i(1-\tilde{p}_i)/n_i}$, and $u_i=\tilde{p}_i+z_{\alpha/2}\sqrt{\tilde{p}_i(1-\tilde{p}_i)/n_i}$. Note that the adjusted Wald method for a single proportion is an approximation of the Wilson score method for 95% interval, see Agresti and Coull (1998) for its motivation and derivation. We also must point out that this method has the potential to provide confidence limits that are out of parameter space.

It is fair to say that the superior performance of Newcombe (1998a) originates from that of the Wilson method for a single proportion (Agresti and Coull, 1998; Newcombe, 1998b). On the same token, we can postulate that applying Wilson interval for cases of more than two binomial proportions will be very competitive to that of Price and Bonett (2004).

To evaluate this claim, we conducted a numerical study to compare the performance of these two procedures in finite samples for 90%, 95%, and 99% two-sided confidence intervals, in terms of mean coverage, minimum coverage, and mean interval width as defined here.

For a $100(1-\alpha)\%$ interval (L,U) for $\sum_{i=1}^{g}c_ip_i$, the coverage is defined by

$$\text{Coverage}=100\sum_{y_1=0}^{n_1}\cdots\sum_{y_g=0}^{n_g}\prod_{i=1}^{g}\binom{n_i}{y_i}p_i^{y_i}(1-p_i)^{n_i-y_i}I\left(L<\sum c_ip_i<U\right),$$

where $I(.)$ is an indicator function which takes values of 1 or 0 as the event in the brackets is true or not.

The expected interval width is defined as

$$\text{Width}=\sum_{y_1=0}^{n_1}\cdots\sum_{y_g=0}^{n_g}\prod_{i=1}^{g}\binom{n_i}{y_i}p_i^{y_i}(1-p_i)^{n_i-y_i}(U-L).$$

We conducted the evaluation by first randomly sampling 1000 sets of p_i's from the uniform (0,1) distribution, and then applied the above two definitions to each set. We did not arbitrarily truncate the adjusted Wald confidence limits when they fell out of the parameter space. With respect to each method, we obtained the mean coverage, minimum coverage, and the mean interval width using these 1000 sets of values for coverage and width.

Table 1
Performance of the method of variance estimates recovery in constructing two-sided confidence intervals (CI) for a linear function of binomial parameters, $\sum_{i=1}^{3} c_i p_i$, using confidence limits for single proportions obtained by the adjusted Wald and Wilson methods. Entries in each row are based on 1000 sets of p_i's randomly sampled from uniform (0,1), and each set evaluated by exact calculation.

Group sizes	90% CI		95% CI	
$n_1/n_2/n_3$	Adjusted Wald	Wilson	Adjusted Wald	Wilson
$c = (1/3, 1/3, 1/3)$				
5/5/5	92.04 (80.18, 0.32)*	90.58 (82.60, 0.31)	96.12 (91.24, 0.38)	94.99 (86.34, 0.36)
5/5/10	91.65 (83.59, 0.29)	90.45 (85.36, 0.29)	95.95 (88.12, 0.35)	95.07 (89.55, 0.33)
5/10/15	91.53 (84.60, 0.25)	90.69 (87.42, 0.25)	95.86 (92.93, 0.30)	95.26 (90.65, 0.29)
5/10/20	91.63 (86.89, 0.25)	90.80 (87.37, 0.24)	95.88 (92.51, 0.29)	95.31 (91.02, 0.28)
5/15/20	91.57 (84.34, 0.23)	90.80 (84.39, 0.23)	95.83 (90.95, 0.28)	95.30 (90.51, 0.27)
5/20/20	91.53 (83.99, 0.23)	90.72 (87.15, 0.22)	95.73 (84.97, 0.27)	95.27 (91.50, 0.26)
$c = (1, -1/2, -1/2)$				
5/5/5	92.29 (80.22, 0.67)	90.82 (85.45, 0.65)	96.19 (86.68, 0.79)	95.14 (89.23, 0.75)
5/5/10	91.97 (84.26, 0.64)	90.84 (84.63, 0.62)	95.91 (89.26, 0.77)	95.27 (89.48, 0.72)
5/10/15	92.00 (82.28, 0.60)	91.06 (86.44, 0.58)	95.87 (86.43, 0.72)	95.31 (91.04, 0.67)
5/10/20	92.00 (82.74, 0.60)	91.06 (85.77, 0.57)	95.83 (87.82, 0.71)	95.33 (91.62, 0.66)
5/15/20	92.00 (82.28, 0.59)	91.06 (86.34, 0.56)	95.75 (88.50, 0.70)	95.27 (91.63, 0.65)
5/20/20	92.13 (81.69, 0.58)	91.13 (85.68, 0.55)	95.80 (86.75, 0.69)	95.27 (91.15, 0.64)
$c = (-1, 1/2, 2)$				
5/5/5	92.08 (79.06, 1.25)	90.94 (86.88, 1.20)	95.91 (86.89, 1.49)	95.19 (89.05, 1.39)
5/5/10	91.52 (85.49, 1.00)	90.67 (86.58, 0.98)	95.81 (91.28, 1.19)	95.24 (89.05, 1.15)
5/10/15	91.33 (85.15, 0.88)	90.64 (87.14, 0.87)	95.66 (91.82, 1.05)	95.29 (88.53, 1.02)
5/10/20	91.35 (85.32, 0.82)	90.72 (87.51, 0.81)	95.71 (92.89, 0.98)	95.32 (90.85, 0.95)
5/15/20	91.29 (82.82, 0.81)	90.65 (87.98, 0.80)	95.67 (92.39, 0.97)	95.23 (91.42, 0.94)
5/20/20	91.29 (85.58, 0.81)	90.65 (86.88, 0.80)	95.66 (90.11, 0.96)	95.30 (90.18, 0.93)
$c = (1, 1, -1)$				
5/5/5	92.04 (80.36, 0.95)	90.56 (81.21, 0.93)	96.19 (92.21, 1.13)	95.15 (86.22, 1.08)
5/5/10	91.74 (85.12, 0.88)	90.64 (85.70, 0.86)	95.96 (89.78, 1.04)	95.18 (89.86, 1.00)
5/10/15	91.49 (84.88, 0.76)	90.71 (87.33, 0.74)	95.78 (87.97, 0.90)	95.26 (90.45, 0.87)
5/10/20	91.49 (85.56, 0.74)	90.76 (86.80, 0.72)	95.80 (92.41, 0.88)	95.29 (90.54, 0.84)
5/15/20	91.42 (85.05, 0.70)	90.69 (86.68, 0.69)	95.70 (90.10, 0.84)	95.20 (91.24, 0.80)
5/20/20	91.59 (85.28, 0.68)	90.82 (87.43, 0.66)	95.82 (88.32, 0.81)	95.26 (90.77, 0.78)

∗ Mean coverage % (minimum coverage %, mean confidence interval width) based on 1000 sets of proportion parameters randomly sampled from uniform (0,1) distribution.

For linear functions of 3 binomial proportions, results in Table 1 show consistently that the intervals for linear functions based on the Wilson score method have mean coverage closer to the nominal levels, with narrow average width. For group sizes considered, the minimum coverage for the adjusted Wald can be as low as 79.06% for 90% nominal level, and 84.97% for 95% nominal level. For confidence interval based on the Wilson method, the minimum coverage can be as low as 81.21% for 90% nominal level, and 86.22% for 95% nominal level. Results from constructing confidence intervals for linear functions of 4 binomial proportions in Table 2 show again that the procedure based on Wilson score method performed better in terms of mean coverage and interval width, as well as minimum coverage. For example, the minimum coverage for the adjusted Wald can be as low as 77.16% for 90% nominal level, compared to that of 83.23% for Wilson score method. Similar trends were observed with nominal level of 99% (results not shown). One possible explanation for our results is that the adjusted Wald method was proposed to approximate the Wilson score method at 95% level, on the rationale that the middle point of Wilson interval is a weighted average of $\widehat{p}$ and 0.5, and that $1.96^2 \approx 4$ (Agresti and Coull, 1998, p. 122).

3. Examples

In the light of the above numerical results, we now compare confidence intervals using two examples from Price and Bonett (2004).

Example 1. This data set arose from a study in which rats are fed with different types of diets. The diets are controlled by two factors, namely fiber and fat. Each rat is observed to determine if it has developed a tumor during the study period. The outcome of the experiment is summarized in Table 3 (each group had 30 rats). It is of interest to construct confidence intervals for the main effects of fiber and fat, as well as their interaction. Here we can obtain the 95% confidence intervals using the MOVER for the linear functions of proportions. The results are shown in Table 3, which shows that the intervals obtained using the Wilson method for single proportions are narrower than those using the adjusted Wald method for single proportion. This is consistent with the results in our evaluation study. In fact, the Wilson method based intervals are all contained in that based on the adjusted Wald method for single proportions in this moderate size study.

Example 2. This example arose from the Framingham heart study. As an alternative to conventional generalized linear model with logistic link function, Price and Bonett (2004) approached the problem with a linear function of binomial

Table 2
Performance of the method of variance estimates recovery in constructing two-sided confidence intervals (CI) for a linear function of binomial parameters, $\sum_{i=1}^{4} c_i p_i$, using confidence limits for single proportions obtained by the adjusted Wald and Wilson methods. Entries in each row are based on 1000 sets of p_i's randomly sampled from uniform (0,1), and each set evaluated by exact calculation.

Group sizes	90% CI		95% CI	
$n_1/n_2/n_3/n_4$	Adjusted Wald	Wilson	Adjusted Wald	Wilson
$c = (1/4, 1/4, 1/4, 1/4)$				
5/5/5/5	91.27 (81.65, 0.28)*	90.24 (83.23, 0.27)	95.63 (92.16, 0.33)	94.86 (88.56, 0.31)
5/5/10/10	91.03 (87.49, 0.24)	90.33 (85.65, 0.24)	95.48 (93.03, 0.29)	95.07 (90.18, 0.28)
5/5/15/15	91.01 (87.61, 0.23)	90.53 (86.71, 0.22)	95.45 (92.91, 0.27)	95.13 (91.33, 0.26)
5/5/15/20	91.11 (88.33, 0.22)	90.67 (87.18, 0.22)	95.50 (93.10, 0.26)	95.25 (91.75, 0.26)
5/10/15/20	90.84 (86.26, 0.20)	90.57 (87.66, 0.20)	95.33 (90.52, 0.24)	95.27 (91.64, 0.23)
$c = (-1, 1, -1, 1)$				
5/5/5/5	91.33 (85.72, 1.10)	90.29 (82.48, 1.08)	95.70 (92.91, 1.31)	95.09 (88.82, 1.25)
5/5/10/10	91.03 (83.19, 0.96)	90.37 (85.96, 0.95)	95.49 (92.68, 1.15)	95.09 (90.56, 1.11)
5/5/15/15	91.11 (87.96, 0.91)	90.67 (86.77, 0.89)	95.50 (92.51, 1.08)	95.24 (91.47, 1.04)
5/5/15/20	91.08 (87.41, 0.89)	90.71 (87.05, 0.88)	95.50 (92.45, 1.06)	95.26 (91.74, 1.02)
5/10/15/20	90.86 (87.58, 0.81)	90.51 (87.68, 0.80)	95.37 (91.86, 0.97)	95.17 (91.40, 0.94)
$c = (1/3, 1/3, 1/3, 1)$				
5/5/5/5	91.33 (77.16, 0.63)	90.87 (86.18, 0.61)	95.29 (84.48, 0.75)	95.15 (89.89, 0.70)
5/5/10/10	90.82 (86.51, 0.49)	90.52 (87.43, 0.49)	95.23 (90.73, 0.59)	95.16 (91.24, 0.57)
5/5/15/15	90.64 (87.56, 0.44)	90.29 (87.33, 0.43)	95.22 (93.25, 0.52)	95.10 (91.60, 0.51)
5/5/15/20	90.74 (88.44, 0.40)	90.25 (86.91, 0.40)	95.34 (93.10, 0.48)	95.07 (90.91, 0.47)
5/10/15/20	90.48 (88.76, 0.39)	90.29 (87.89, 0.38)	95.12 (93.65, 0.46)	95.15 (92.17, 0.45)
$c = (-3, -1, 1, 3)$				
5/5/5/5	91.34 (83.03, 2.44)	90.89 (83.44, 2.38)	95.49 (90.28, 2.90)	95.20 (89.81, 2.76)
5/5/10/10	90.94 (83.95, 2.14)	90.80 (87.55, 2.10)	95.23 (88.85, 2.55)	95.32 (91.50, 2.44)
5/5/15/15	91.01 (82.16, 2.01)	90.80 (87.95, 1.96)	95.24 (86.35, 2.40)	95.22 (90.66, 2.29)
5/5/15/20	90.92 (83.77, 1.96)	90.77 (88.24, 1.90)	95.13 (86.91, 2.33)	95.19 (91.50, 2.22)
5/10/15/20	91.08 (83.32, 1.91)	91.02 (88.56, 1.86)	95.15 (85.88, 2.27)	95.36 (91.06, 2.16)

* Mean coverage % (minimum coverage %, mean confidence interval width) based on 1000 sets of proportion parameters randomly sampled from uniform (0,1) distribution.

Table 3
Confidence intervals for effects of factors in the diet–tumor study.

Fiber	Fat	$\widehat{p}_i$	c_i		
			Fiber × Fat	Fiber	Fat
Yes	High	20/30	1	1/2	1/2
	Low	14/30	−1	1/2	−1/2
No	High	27/30	1	−1/2	1/2
	Low	19/30	−1	−1/2	−1/2
Interval for $\sum c_i p_i$:					
		Adj Wald	−0.3806, 0.2516	−0.3516, 0.0355	0.0677, 0.3839
		Wilson	−0.3790, 0.2386	−0.3459, 0.0375	0.0691, 0.3773

Table 4
Framingham heart study.

Systolic BP	Number of subjects	Number with heart disease
115	156	3
121	252	17
131	284	12
141	271	16
151	139	12
161	85	8
176	99	16
190	43	8

proportions. Specifically, if the population proportion of heart disease is considered a linear function of systolic blood pressure, the slope is $\sum c_i p_i$, which is a linear function of the proportions p_i of heart disease of systolic blood pressure groups, where $c_i = (x_i - \sum x_i/g)/\sum(x_i - \sum x_i/g)^2$ and x_i is the value of the quantitative factor in group i. Using the data in Table 4, we obtained the 95% confidence interval for the population slope using the adjusted Wald method as 0.0010 to 0.0032, comparable to that of using the Wilson method as 0.0012 to 0.0034 in such a large study.

4. Concluding remarks

The confidence interval for a general linear function of binomial proportions introduced here is a simple application of a more general idea presented by Zou and Donner (2008). The basic idea is to recover variance estimates needed for linear functions of proportions from the confidence limits for single proportions. Since the Wilson interval procedure has been strongly recommended for single proportions (Agresti and Coull, 1998; Newcombe, 1998b; Santner, 1998), it is thus natural to extend it to linear functions of binomial proportions. By use of the MOVER, we have provided a very competitive procedure to that of Price and Bonett (2004), whose procedure can be seen as an application of the MOVER based on the adjusted Wald method for single proportions. The MOVER has also provided an analytic justification for Newcombe (1998a, 2001).

It should also be noted that the derivation of the MOVER relies only on the validity of confidence limits for single parameters such that variance estimates can be recovered by normal distributions. The direct implication is that one can apply the MOVER to linear functions of other discrete distribution parameters, e.g., Poisson rates (Stamey and Hamilton, 2006; Tebbs and Roths, 2008), and linear functions of normal mean and variance, e.g., lognormal means (Zou and Donner, 2008).

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References

Agresti, A., Coull, B., 1998. Approximate is better than "exact" for interval estimation of binomial proportions. American Statistician 52, 119–126.
Bartlett, M.S., 1953. Approximate confidence intervals. 2. More than one unknown parameter. Biometrika 40, 306–317.
Efron, B., Tibshirani, R.J., 1993. An Introduction to the Bootstrap. Chapman & Hall/CRC, New York.
Gart, J.J., Nam, J.M., 1990. Approximate interval estimation of the difference in binomial parameters - correction for skewness and extension to multiple tables. Biometrics 46, 637–643.
Howe, W.G., 1974. Approximate confidence limits on the mean of $X + Y$ where X and Y are two tabled independent random variables. Journal of the American Statistical Association 69, 789–794.
Mee, R.W., 1984. Confidence bounds for the difference between two probabilities. Biometrics 40, 1175–1176.
Miettinen, O., Nurminen, M., 1985. Comparative analysis of two rates. Statistics in Medicine 4, 213–226.
Newcombe, R.G., 1998a. Interval estimation for the difference between independent proportions: Comparison of eleven methods. Statistics in Medicine 17, 873–890.
Newcombe, R.G., 1998b. Two-sided confidence intervals for the single proportions: Comparison of seven methods. Statistics in Medicine 17, 857–872.
Newcombe, R.G., 2001. Estimating the difference between differences: Measurement of additive scale interaction for proportions. Statistics in Medicine 20, 2885–2893.
Price, R.M., Bonett, D.G., 2004. An improved confidence interval for a linear function of binomial proportions. Computational Statistics & Data Analysis 45, 449–456.
Santner, T.J., 1998. Teaching large-sample binomial confidence intervals. Teaching Statistics 20, 20–23.
Stamey, J., Hamilton, C., 2006. A note on confidence intervals for a linear function of Poisson rates. Communications in Statistics–Simulation and Computation 35, 849–856.
Tebbs, J.M., Roths, S.A., 2008. New large-sample confidence intervals for a linear combination of binomial proportions. Journal of Statistical Planning and Inference 138, 1884–1893.
Wilson, E.B., 1927. Probable inference, the law of succession, and statistical inference. Journal of the American Statistical Association 22, 209–212.
Zou, G.Y., 2008. On the estimation of additive interaction using the four-by-two table and beyond. American Journal of Epidemiology 168, 212–224.
Zou, G.Y., Donner, A., 2008. Construction of confidence limits about effect measures: A general approach. Statistics in Medicine 27, 1693–1702.