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Confidence Intervals for Mean and Difference of Means of Normal Distributions with Unknown Coefficients of Variation

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Abstract: This paper proposes confidence intervals for a single mean and difference of two means of normal distributions with unknown coefficients of variation (CVs). The generalized confidence interval (GCI) approach and large sample (LS) approach were proposed to construct confidence intervals for the single normal mean with unknown CV. These confidence intervals were compared with existing confidence interval for the single normal mean based on the Student's t-distribution (small sample size case) and the z-distribution (large sample size case). Furthermore, the confidence intervals for the difference between two normal means with unknown CVs were constructed based on the GCI approach, the method of variance estimates recovery (MOVER) approach and the LS approach and then compared with the Welch–Satterthwaite (WS) approach. The coverage probability and average length of the proposed confidence intervals were evaluated via Monte Carlo simulation. The results indicated that the GCIs for the single normal mean and the difference of two normal means with unknown CVs are better than the other confidence intervals. Finally, three datasets are given to illustrate the proposed confidence intervals.

Keywords: mean; coefficient of variation (CV); normal distribution; generalized confidence interval (GCI) approach; method of variance estimates recovery (MOVER) approach

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1. Introduction

It is well known that the sample mean, $\bar{x}$, is the uniformly minimum variance unbiased (UMVU) estimator of the normal population mean μ; see the paper by Sahai et al. [1]. Dropping the requirement of unbiasedness, Searls [2] proposed the minimum mean squared error (MMSE) estimator for normal mean with known coefficient of variation (CV). Khan [3] discussed the estimation of the mean with known CV in one sample case. Gleser and Healy [4] proposed the minimum quadratic risk scale-invariant estimator for the normal mean with known CV. Bhat and Rao [5] investigated the tests for a normal mean with known CV. Niwitpong et al. [6] provided confidence intervals for the difference between normal population means with known CVs. Niwitpong [7] presented confidence intervals for the normal mean with known CV. Niwitpong and Niwitpong [8] proposed the confidence interval for the normal mean with a known CV based on the best unbiased estimator, which was proposed by Khan [3]. Niwitpong [9] proposed the confidence interval for the normal mean with a known CV based on the t-test. Niwitpong and Niwitpong [10] constructed new confidence intervals for the difference between normal means with known CV. Sodanin et al. [11] proposed confidence intervals for the common mean of normal distributions with known CV.

In practice, the CV is unknown. Furthermore, the CV needs to be estimated. Therefore, Srivastava [12] proposed a UMVU estimator for the estimation of the normal mean with unknown CV,

$\hat{\theta} = \bar{x}/(1+(s^2/(n\bar{x}^2)))$, where CV is defined as σ/μ. The UMVU estimator, estimated from the MMSE estimator of Searls [2], is more efficient than the usual unbiased estimator sample mean $\bar{x}$ whenever $\sigma^2/(\mu\sigma^2)$ is at least 0.5. Srivastava and Singh [13] provided a UMVU estimate of the relative efficiency ratio of $\hat{\theta}$. Moreover, Sahai [14] developed a new estimator for the normal mean with unknown CV. Sahai and Acharya [15] studied the iterative estimation of the normal population mean using computational-statistical intelligence. However, a confidence interval provides more information about a population value of the quantity than a point estimate. Therefore, it is of practical and theoretical importance to develop procedures for confidence interval estimation of the mean of the normal distribution with unknown CV. Hence, along similar lines as Srivastava [12], we construct the new confidence intervals for the normal mean with unknown CV and compare with the standard confidence intervals: the Student's t-distribution and the z-distribution. The comparison can be based on coverage probability, as well as the length of the confidence intervals. The average length of the confidence intervals could also be analytically obtained and hence compared; see, e.g., Sodanin et al. [16], who proposed the confidence intervals for the normal population mean with unknown CV based on the generalized confidence interval (GCI) approach. This paper extends the work of Sodanin et al. [16] to construct confidence intervals for the normal population mean with unknown CV based on the GCI approach and the new confidence intervals based on the large sample (LS) approach. Furthermore, three new confidence intervals for the difference between normal means with unknown CVs were also proposed based on the GCI approach, the LS approach and the method of variance estimates recovery (MOVER) approach and compared with the well-known Welch–Satterthwaite (WS) approach. For more on confidence intervals on CV, we refer our readers to Banik and Kibria [17], Gulhar et al. [18] and, recently, Albatineh et al. [19], among others.

This paper is organized as follows. In Section 2, the confidence intervals for the single normal mean with unknown CV are presented. In Section 3, the confidence intervals for the difference between normal means with unknown CVs are provided. In Section 4, simulation results are presented to evaluate the coverage probabilities and average lengths in the comparison of the proposed approaches. In Section 5, the proposed approaches are illustrated using three examples. Section 6 summarizes this paper.

2. Confidence Intervals for the Mean of the Normal Distribution with Unknown Coefficient of Variation

Suppose that $X = (X_1, X_2, \ldots, X_n)$ are independent random variables each having the normal distribution with mean μ and variance σ^2. The CV is defined by $\tau = \sigma/\mu$. Let $\bar{X}$ and S^2 be the sample mean and sample variance for X, respectively. Furthermore, let $\bar{x}$ and s^2 be the observed sample of $\bar{X}$ and S^2, respectively.

Searls [2] proposed the MMSE estimator for the normal population mean with variance, θ, defined by:

$$\theta = \frac{\mu}{1+(\sigma^2/n\mu^2)} = \frac{n\mu}{n+(\sigma^2/\mu^2)}. \quad (1)$$

However, the CV needs to be estimated. Srivastava [12] proposed an estimator of the mean with unknown CV, which is defined by:

$$\hat{\theta} = \frac{\bar{X}}{1+(S^2/n\bar{X}^2)} = \frac{n\bar{X}}{n+(S^2/\bar{X}^2)}. \quad (2)$$

Moreover, Sahai [14] proposed an alternative estimator of the normal population mean with unknown CV, which is defined by:

$$\theta^* = \frac{\mu}{1-(\sigma^2/n\mu^2)} = \frac{n\mu}{n-(\sigma^2/\mu^2)}. \quad (3)$$

The estimator of θ^* is defined by:

$$\hat{\theta}^* = \frac{\bar{X}}{1 - (S^2/n\bar{X}^2)} = \frac{n\bar{X}}{n - (S^2/\bar{X}^2)}. \quad (4)$$

Theorem 1. *Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$. Suppose $\bar{X}$ and S^2 are a sample mean and a sample variance, respectively. Let θ^* be an estimator of the normal population mean with unknown CV, and let $\hat{\theta}^*$ be an estimator of θ^*. The mean and variance of $\hat{\theta}^*$ are obtained by:*

$$E\left(\hat{\theta}^*\right) = \left(\frac{\mu}{1 - \left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)\left(1 + \frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n} + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n - \left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right) \quad (5)$$

and:

$$Var\left(\hat{\theta}^*\right) = \left(\frac{\mu}{1 - \left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)^2\left(\frac{\sigma^2}{n\mu^2} + \frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n} + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n - \left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right). \quad (6)$$

Proof. Let $\theta^* = n\mu/(n - (\sigma^2/\mu^2))$ and $\hat{\theta}^* = n\bar{X}/(n - (S^2/\bar{X}^2))$. Since $\bar{X} \sim N(\mu, \sigma^2/n)$. Then:

$$E\left(n\bar{X}\right) = nE\left(\bar{X}\right) = n\mu \quad \text{and} \quad Var\left(n\bar{X}\right) = n^2 Var\left(\bar{X}\right) = \frac{n^2\sigma^2}{n} = n\sigma^2.$$

According to Thangjai et al. [20], the mean of $\bar{X}^2$ is computed by the moment generating function, and the variance of $\bar{X}^2$ is computed by Stein's lemma. Therefore, the mean and the variance of $\bar{X}^2$ are defined by:

$$E\left(\bar{X}^2\right) = \frac{n\mu^2 + \sigma^2}{n} \quad \text{and} \quad Var\left(\bar{X}^2\right) = \frac{2\sigma^4 + 4n\mu^2\sigma^2}{n^2}.$$

From Thangjai et al. [20], the mean and the variance of $S^2/\bar{X}^2$ are defined by:

$$E\left(\frac{S^2}{\bar{X}^2}\right) = \left(\frac{n\sigma^2}{n\mu^2 + \sigma^2}\right)\left(1 + \frac{2\sigma^4 + 4n\mu^2\sigma^2}{(n\mu^2 + \sigma^2)^2}\right)$$

and:

$$Var\left(\frac{S^2}{\bar{X}^2}\right) = \left(\frac{n\sigma^2}{n\mu^2 + \sigma^2}\right)^2\left(\frac{2}{n} + \frac{2\sigma^4 + 4n\mu^2\sigma^2}{(n\mu^2 + \sigma^2)^2}\right).$$

Therefore, the mean and variance of $n - (S^2/\bar{X}^2)$ are defined by:

$$E\left(n - \frac{S^2}{\bar{X}^2}\right) = n - \left(\frac{n\sigma^2}{n\mu^2 + \sigma^2}\right)\left(1 + \frac{2\sigma^4 + 4n\mu^2\sigma^2}{(n\mu^2 + \sigma^2)^2}\right)$$

and:

$$Var\left(n - \frac{S^2}{\bar{X}^2}\right) = Var\left(\frac{S^2}{\bar{X}^2}\right) = \left(\frac{n\sigma^2}{n\mu^2 + \sigma^2}\right)^2\left(\frac{2}{n} + \frac{2\sigma^4 + 4n\mu^2\sigma^2}{(n\mu^2 + \sigma^2)^2}\right).$$

According to Blumenfeld [21], the mean and variance of $\hat{\theta}^*$ are obtained by:

$$\begin{aligned}
E\left(\hat{\theta}^*\right) &= E\left(\frac{n\bar{X}}{n-(S^2/\bar{X}^2)}\right) \\
&= \left(\frac{E\left(n\bar{X}\right)}{E\left(n-(S^2/\bar{X}^2)\right)}\right)\left(1+\frac{Var\left(n-(S^2/\bar{X}^2)\right)}{\left(E\left(n-(S^2/\bar{X}^2)\right)\right)^2}\right) \\
&= \left(\frac{n\mu}{n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)\left(1+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right) \\
&= \left(\frac{\mu}{1-\left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)\left(1+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right)
\end{aligned}$$

and:

$$\begin{aligned}
Var\left(\hat{\theta}^*\right) &= Var\left(\frac{n\bar{X}}{n-(S^2/\bar{X}^2)}\right) \\
&= \left(\frac{E\left(n\bar{X}\right)}{E\left(n-(S^2/\bar{X}^2)\right)}\right)^2\left(\frac{Var\left(n\bar{X}\right)}{\left(E\left(n\bar{X}\right)\right)^2}+\frac{Var\left(n-(S^2/\bar{X}^2)\right)}{\left(E\left(n-(S^2/\bar{X}^2)\right)\right)^2}\right) \\
&= \left(\frac{n\mu}{n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)^2\left(\frac{n\sigma^2}{n^2\mu^2}+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right) \\
&= \left(\frac{\mu}{1-\left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)^2\left(\frac{\sigma^2}{n\mu^2}+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right).
\end{aligned}$$

Hence, Theorem 1 is proven. □

Proposition 1. *Let $X=(X_1,X_2,\ldots,X_n)$ be a random sample from the normal distribution with the mean μ and the variance σ^2. Let $\bar{X}$ and S^2 be the corresponding point estimates of μ and σ^2. Then:*

$$\sqrt{n}\left(\hat{\theta}^*-\tau_1^*\right)\approx\sqrt{n}\left(\hat{\theta}^*-\mu\right)\xrightarrow{D}N\left(0,n\tau_2^*\right), \tag{7}$$

where $\hat{\theta}^*=n\bar{X}/(n-(S^2/\bar{X}^2))$, τ_1^* is $E\left(\hat{\theta}^*\right)$ in Equation (5) and τ_2^* is $Var\left(\hat{\theta}^*\right)$ in Equation (6).

Proof. Let $\theta^*=n\mu/(n-(\sigma^2/\mu^2))$ be an estimator of the mean with unknown CV, and let $\hat{\theta}^*=n\bar{X}/(n-(S^2/\bar{X}^2))$ be an estimator of θ^*. From Theorem 1, $\hat{\theta}^*$ is distributed normally with mean τ_1^* and variance τ_2^*, which is defined by:

$$\hat{\theta}^*\sim N(\tau_1^*,\tau_2^*),$$

where:

$$\tau_1^*=\left(\frac{\mu}{1-\left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)\left(1+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right)$$

and:

$$\tau_2^* = \left(\frac{\mu}{1 - \left(\frac{\sigma^2}{n\mu^2+\sigma^2} \right) \left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right)} \right)^2 \left(\frac{\sigma^2}{n\mu^2} + \frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2} \right)^2 \left(\frac{2}{n} + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right)}{\left(n - \left(\frac{n\sigma^2}{n\mu^2+\sigma^2} \right) \left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right) \right)^2} \right).$$

Applying the asymptotic theory, the estimator $\hat{\theta}^*$ is consistent. That is, $\hat{\theta}^*$ converges in probability to τ_1^* and τ_1^* converges in probability to μ as $n \longrightarrow \infty$. The estimator is asymptotically normal and is defined by:

$$\sqrt{n}\left(\hat{\theta}^* - \tau_1^*\right) \approx \sqrt{n}\left(\hat{\theta}^* - \mu\right) \xrightarrow{D} N\left(0, n\tau_2^*\right),$$

where $\xrightarrow{D}$ represents that it converges in the distribution. Hence, Proposition 1 is proven. □

Theorem 2. *Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a random sample from $N(\mu, \sigma^2)$. Let θ be an estimator of normal population mean with unknown CV, and let $\hat{\theta}$ be an estimator of θ. The mean and variance of $\hat{\theta}$ are obtained by:*

$$E\left(\hat{\theta}\right) = \left(\frac{\mu}{1 + \left(\frac{\sigma^2}{n\mu^2+\sigma^2} \right) \left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right)} \right) \left(1 + \frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2} \right)^2 \left(\frac{2}{n} + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right)}{\left(n + \left(\frac{n\sigma^2}{n\mu^2+\sigma^2} \right) \left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right) \right)^2} \right) \quad (8)$$

and:

$$Var\left(\hat{\theta}\right) = \left(\frac{\mu}{1 + \left(\frac{\sigma^2}{n\mu^2+\sigma^2} \right) \left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right)} \right)^2 \left(\frac{\sigma^2}{n\mu^2} + \frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2} \right)^2 \left(\frac{2}{n} + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right)}{\left(n + \left(\frac{n\sigma^2}{n\mu^2+\sigma^2} \right) \left(1 + \frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2} \right) \right)^2} \right). \quad (9)$$

Proof. For the proof of the mean and variance of $\hat{\theta}$ is similarly to Theorem 1. □

Proposition 2. *Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from the normal distribution with the mean μ and the variance σ^2. Let $\bar{X}$ and S^2 be the corresponding point estimates of μ and σ^2. Then:*

$$\sqrt{n}\left(\hat{\theta} - \tau_1\right) \approx \sqrt{n}\left(\hat{\theta} - \mu\right) \xrightarrow{D} N\left(0, n\tau_2\right), \quad (10)$$

where $\hat{\theta} = n\bar{X}/(n + (S^2/\bar{X}^2))$, τ_1 is $E\left(\hat{\theta}\right)$ in Equation (8), and τ_2 is $Var\left(\hat{\theta}\right)$ in Equation (9).

Proof. For the proof of the distribution of $\hat{\theta}$ is similar to Proposition 1. □

2.1. Generalized Confidence Intervals for the Mean of the Normal Distribution with Unknown Coefficient of Variation

Definition 1. *Let $X = (X_1, X_2, \ldots, X_n)$ be a random sample from a distribution $F(x|\delta)$, which depends on a vector of parameters $\delta = (\theta, \vartheta)$ where θ is the parameter of interest and ϑ is possibly a vector of nuisance parameters. Weerahandi [22] defines a generalized pivot $R(X, x, \theta, \vartheta)$ for confidence interval estimation, where x is an observed value of X, as a random variable having the following two properties:*

(i) $R(X, x, \theta, \vartheta)$ has a probability distribution that is free of unknown parameters.
(ii) The observed value of $R(X, x, \theta, \vartheta)$, $X = x$, is the parameter of interest.

Let $R(\alpha)$ be the 100α-th percentile of $R(X, x, \theta, \vartheta)$. Then, $(R(\alpha/2), R(1-\alpha/2))$ becomes a $100(1-\alpha)\%$ two-sided GCI for θ.

Recall that:

$$\frac{(n-1)\,S^2}{\sigma^2} = V \sim \chi^2_{n-1}, \tag{11}$$

where V is chi-squared distribution with $n-1$ degrees of freedom. Now, write:

$$\sigma^2 = \frac{(n-1)\,S^2}{V}. \tag{12}$$

The generalized pivotal quantity (GPQ) for σ^2 is defined by:

$$R_{\sigma^2} = \frac{(n-1)\,s^2}{V}. \tag{13}$$

Moreover, the mean is given by:

$$\mu \approx \bar{x} - \frac{Z}{\sqrt{U}}\sqrt{\frac{(n-1)\,s^2}{n}}, \tag{14}$$

where Z and U denote the standard normal distribution and chi-square distribution with $n-1$ degrees of freedom, respectively. Thus, the GPQ for μ is defined by:

$$R_{\mu} = \bar{x} - \frac{Z}{\sqrt{U}}\sqrt{\frac{(n-1)\,s^2}{n}}. \tag{15}$$

Therefore, the GPQ for θ is defined by:

$$R_{\theta} = \frac{nR_{\mu}}{n + \left(R_{\sigma^2}/R^2_{\mu}\right)}. \tag{16}$$

Moreover, the GPQ for θ^* is defined by:

$$R_{\theta^*} = \frac{nR_{\mu}}{n - \left(R_{\sigma^2}/R^2_{\mu}\right)}. \tag{17}$$

Therefore, the $100\,(1-\alpha)\,\%$ two-sided confidence intervals for the single normal mean with unknown CV based on the GCI approach are obtained by:

$$CI_{GCI.\theta} = \left(R_{\theta}\,(\alpha/2)\,, R_{\theta}\,(1-\alpha/2)\right) \tag{18}$$

and:

$$CI_{GCI.\theta^*} = \left(R_{\theta^*}\,(\alpha/2)\,, R_{\theta^*}\,(1-\alpha/2)\right), \tag{19}$$

where $R_{\theta}\,(\alpha)$ and $R_{\theta^*}\,(\alpha)$ denote the $100\,(\alpha)$-th percentiles of R_{θ} and R_{θ^*}, respectively.

Algorithm 1. For a given $\bar{x}$, the GCI for θ and θ^* can be computed by the following steps:

Step 1. Generate $V \sim \chi^2_{n-1}$, and then, compute R_{σ^2} from Equation (13).
Step 2. Generate $Z \sim N(0,1)$ and $U \sim \chi^2_{n-1}$, then compute R_{μ} from Equation (15).
Step 3. Compute R_{θ} from Equation (16), and compute R_{θ^*} from Equation (17).
Step 4. Repeat Steps 1–3 a total q times, and obtain an array of R_{θ}'s and R_{θ^*}'s.
Step 5. Compute $R_{\theta}\,(\alpha/2)$, $R_{\theta}\,(1-\alpha/2)$, $R_{\theta^*}\,(\alpha/2)$ and $R_{\theta^*}\,(1-\alpha/2)$.

2.2. Large Sample Confidence Intervals for the Mean of the Normal Distribution with Unknown Coefficient of Variation

Again, from Equations (2) and (4), the estimators of the mean with unknown CV are defined by:

$$\hat{\theta} = \frac{n\bar{X}}{n + (S^2/\bar{X}^2)} \tag{20}$$

and:

$$\hat{\theta}^* = \frac{n\bar{X}}{n - (S^2/\bar{X}^2)}. \tag{21}$$

From Theorem 2, the variance of $\hat{\theta}$ is defined by:

$$Var\left(\hat{\theta}\right) = \left(\frac{\mu}{1+\left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)^2\left(\frac{\sigma^2}{n\mu^2}+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n+\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right), \tag{22}$$

with μ and σ^2 replaced by $\bar{x}$ and s^2, respectively.

From Theorem 1, the variance of $\hat{\theta^*}$ is defined by:

$$Var\left(\hat{\theta}^*\right) = \left(\frac{\mu}{1-\left(\frac{\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}\right)^2\left(\frac{\sigma^2}{n\mu^2}+\frac{\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)}{\left(n-\left(\frac{n\sigma^2}{n\mu^2+\sigma^2}\right)\left(1+\frac{2\sigma^4+4n\mu^2\sigma^2}{(n\mu^2+\sigma^2)^2}\right)\right)^2}\right), \tag{23}$$

with μ and σ^2 replaced by $\bar{x}$ and s^2, respectively.

Therefore, the $100\,(1-\alpha)\,\%$ two-sided confidence intervals for the single normal mean with unknown CV based on the LS approach are obtained by:

$$CI_{LS.\theta} = \left(\hat{\theta} - z_{1-\alpha/2}\sqrt{Var(\hat{\theta})}, \hat{\theta} + z_{1-\alpha/2}\sqrt{Var(\hat{\theta})}\right) \tag{24}$$

and:

$$CI_{LS.\theta^*} = \left(\hat{\theta^*} - z_{1-\alpha/2}\sqrt{Var(\hat{\theta^*})}, \hat{\theta^*} + z_{1-\alpha/2}\sqrt{Var(\hat{\theta^*})}\right), \tag{25}$$

where $z_{1-\alpha/2}$ denotes the $1-\alpha/2$-th quantile of the standard normal distribution.

Algorithm 2. The coverage probability for θ and θ^* can be computed by the following steps:

Step 1. Generate $X_1, X_2, \ldots, X_n$ from $N\left(\mu, \sigma^2\right)$ and then compute $\bar{x}$ and s^2.
Step 2. Use Algorithm 1 to construct $CI_{GCI.\theta}$ and record whether or not the value of θ falls in the corresponding confidence interval.
Step 3. Use Algorithm 1 to construct $CI_{GCI.\theta^*}$ and record whether or not the value of θ^* falls in the corresponding confidence interval.
Step 4. Use Equation (24) to construct $CI_{LS.\theta}$ and record whether or not the value of θ falls in the corresponding confidence interval.
Step 5. Use Equation (25) to construct $CI_{LS.\theta^*}$ and record whether or not the value of θ^* falls in the corresponding confidence interval.
Step 6. Repeat Steps 1–5, a total M times. Then, for $CI_{GCI.\theta}$ and $CI_{LS.\theta}$, the fraction of times that all θ are in their corresponding confidence intervals provides an estimate of the coverage probability. Similarly, for $CI_{GCI.\theta^*}$ and $CI_{LS.\theta^*}$, the fraction of times that all θ^* are in their corresponding confidence intervals provides an estimate of the coverage probability.

3. Confidence Intervals for the Difference between the Means of Normal Distributions with Unknown Coefficients of Variation

Suppose that $X = (X_1, X_2, \ldots, X_n)$ are independent random variables each having a normal distribution with mean μ_X and variance σ_X^2. Additionally, suppose that $Y = (Y_1, Y_2, \ldots, Y_m)$ are independent random variables each having a normal distribution with mean μ_Y and variance σ_Y^2. Furthermore, X and Y are independent. Let $\bar{X}$ and S_X^2 be the sample mean and the sample variance for X, respectively. Furthermore, let $\bar{x}$ and s_X^2 be the observed sample of $\bar{X}$ and S_X^2, respectively. Similarly, let $\bar{Y}$ and S_Y^2 be the sample mean and the sample variance for Y, respectively. Furthermore, let $\bar{y}$ and s_Y^2 be the observed sample of $\bar{Y}$ and S_Y^2, respectively.

Let $\delta = \theta_X - \theta_Y$ be the difference between means with unknown CVs. The estimators of δ are defined by:

$$\hat{\delta} = \hat{\theta}_X - \hat{\theta}_Y = \frac{n\bar{X}}{n + (S_X^2/\bar{X}^2)} - \frac{m\bar{Y}}{m + (S_Y^2/\bar{Y}^2)} \tag{26}$$

and:

$$\hat{\delta}^* = \hat{\theta}_X^* - \hat{\theta}_Y^* = \frac{n\bar{X}}{n - (S_X^2/\bar{X}^2)} - \frac{m\bar{Y}}{m - (S_Y^2/\bar{Y}^2)}, \tag{27}$$

where $\hat{\theta}_X$ and $\hat{\theta}_X^*$ denote the estimator of θ_X and θ_X^*, respectively, and $\hat{\theta}_Y$ and $\hat{\theta}_Y^*$ denote the estimator of θ_Y and θ_Y^*, respectively.

Theorem 3. *Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a random sample from $N\left(\mu_X, \sigma_X^2\right)$, and suppose that $Y = (Y_1, Y_2, \ldots, Y_m)$ is a random sample from $N\left(\mu_Y, \sigma_Y^2\right)$. Let X and Y be independent. Let $\bar{X}$ and S_X^2 be the sample mean and the sample variance for X, respectively. Furthermore, let $\bar{Y}$ and S_Y^2 be the sample mean and the sample variance for Y, respectively. Let θ_X and θ_Y be the mean with unknown CV of X and Y, respectively. Let δ be the difference between θ_X and θ_Y. Let $\hat{\delta}$ be an estimator of δ. The mean and variance of $\hat{\delta}$ are obtained by:*

$$\begin{aligned} E\left(\hat{\delta}\right) = {} & \left(\frac{\mu_X}{1 + \left(\frac{\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1 + \frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}\right)\left(1 + \frac{\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)^2\left(\frac{2}{n} + \frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}{\left(n + \left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1 + \frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)\right)^2}\right) \\ & - \left(\frac{\mu_Y}{1 + \left(\frac{\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1 + \frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}\right)\left(1 + \frac{\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)^2\left(\frac{2}{m} + \frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}{\left(m + \left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1 + \frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)\right)^2}\right) \end{aligned} \tag{28}$$

and:

$$\begin{aligned} Var\left(\hat{\delta}\right) = {} & \left(\frac{\mu_X}{1+\left(\frac{\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}\right)^2\left(\frac{\sigma_X^2}{n\mu_X^2} + \frac{\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}{\left(n+\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)\right)^2}\right) \\ & + \left(\frac{\mu_Y}{1+\left(\frac{\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}\right)^2\left(\frac{\sigma_Y^2}{m\mu_Y^2} + \frac{\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)^2\left(\frac{2}{m}+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}{\left(m+\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)\right)^2}\right). \end{aligned} \tag{29}$$

Proof. Let $\delta = \theta_X - \theta_Y$ be the difference between means with unknown CVs. Let $\hat{\delta}$ be an estimator of δ, which is defined by:

$$\hat{\delta} = \frac{n\bar{X}}{n + (S_X^2/\bar{X}^2)} - \frac{m\bar{Y}}{m + (S_Y^2/\bar{Y}^2)}.$$

Thus, the mean and variance of $\hat{\delta}$ are obtained by:

$$\begin{aligned}
E\left(\hat{\delta}\right) &= E\left(\frac{n\bar{X}}{n+(S_X^2/\bar{X}^2)} - \frac{m\bar{Y}}{m+(S_Y^2/\bar{Y}^2)}\right) \\
&= E\left(\frac{n\bar{X}}{n+(S_X^2/\bar{X}^2)}\right) - E\left(\frac{m\bar{Y}}{m+(S_Y^2/\bar{Y}^2)}\right) \\
&= \left(\frac{\mu_X}{1+\left(\frac{\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}\right)\left(1+\frac{\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}{\left(n+\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)\right)^2}\right) \\
&- \left(\frac{\mu_Y}{1+\left(\frac{\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}\right)\left(1+\frac{\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)^2\left(\frac{2}{m}+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}{\left(m+\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)\right)^2}\right)
\end{aligned}$$

and:

$$\begin{aligned}
Var\left(\hat{\delta}\right) &= Var\left(\frac{n\bar{X}}{n+(S_X^2/\bar{X}^2)} - \frac{m\bar{Y}}{m+(S_Y^2/\bar{Y}^2)}\right) \\
&= Var\left(\frac{n\bar{X}}{n+(S_X^2/\bar{X}^2)}\right) + Var\left(\frac{m\bar{Y}}{m+(S_Y^2/\bar{Y}^2)}\right) \\
&= \left(\frac{\mu_X}{1+\left(\frac{\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}\right)^2\left(\frac{\sigma_X^2}{n\mu_X^2}+\frac{\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}{\left(n+\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)\right)^2}\right) \\
&+ \left(\frac{\mu_Y}{1+\left(\frac{\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}\right)^2\left(\frac{\sigma_Y^2}{m\mu_Y^2}+\frac{\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)^2\left(\frac{2}{m}+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}{\left(m+\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)\right)^2}\right).
\end{aligned}$$

Hence, Theorem 3 is proven. □

Theorem 4. *Suppose that $X = (X_1, X_2, \ldots, X_n)$ is a random sample from $N\left(\mu_X, \sigma_X^2\right)$ and suppose that $Y = (Y_1, Y_2, \ldots, Y_m)$ is a random sample from $N\left(\mu_Y, \sigma_Y^2\right)$. Let X and Y be independent. Let $\bar{X}$ and S_X^2 be the sample mean and the sample variance for X, respectively. Furthermore, let $\bar{Y}$ and S_Y^2 be the sample mean and the sample variance for Y, respectively. Let θ_X^* and θ_Y^* be the mean with unknown CV of X and Y, respectively. Let δ^* be the difference between θ_X^* and θ_Y^*. Additionally, let $\hat{\delta}^*$ be an estimator of δ^*. The mean and variance of $\hat{\delta}^*$ are obtained by:*

$$\begin{aligned}
E\left(\hat{\delta}^*\right) = &\left(\frac{\mu_X}{1-\left(\frac{\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}\right)\left(1+\frac{\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}{\left(n-\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)\right)^2}\right) \\
&- \left(\frac{\mu_Y}{1-\left(\frac{\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}\right)\left(1+\frac{\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)^2\left(\frac{2}{m}+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}{\left(m-\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)\right)^2}\right) \qquad (30)
\end{aligned}$$

and:

$$Var\left(\hat{\delta}^{*}\right)=\left(\frac{\mu_X}{1-\left(\frac{\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}\right)^2\left(\frac{\sigma_X^2}{n\mu_X^2}+\frac{\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)^2\left(\frac{2}{n}+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)}{\left(n-\left(\frac{n\sigma_X^2}{n\mu_X^2+\sigma_X^2}\right)\left(1+\frac{2\sigma_X^4+4n\mu_X^2\sigma_X^2}{\left(n\mu_X^2+\sigma_X^2\right)^2}\right)\right)^2}\right)$$

$$+\left(\frac{\mu_Y}{1-\left(\frac{\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}\right)^2\left(\frac{\sigma_Y^2}{m\mu_Y^2}+\frac{\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)^2\left(\frac{2}{m}+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)}{\left(m-\left(\frac{m\sigma_Y^2}{m\mu_Y^2+\sigma_Y^2}\right)\left(1+\frac{2\sigma_Y^4+4m\mu_Y^2\sigma_Y^2}{\left(m\mu_Y^2+\sigma_Y^2\right)^2}\right)\right)^2}\right). \quad (31)$$

Proof. For the proof of the mean and variance of $\hat{\delta}^{*}$ is similar to Theorem 3. □

3.1. Generalized Confidence Intervals for the Difference between Means of Normal Distributions with Unknown Coefficients of Variation

From the random variable X and Y, since:

$$\frac{(n-1)\,S_X^2}{\sigma_X^2}=V_X\sim\chi_{n-1}^2 \text{ and } \frac{(m-1)\,S_Y^2}{\sigma_Y^2}=V_Y\sim\chi_{m-1}^2. \quad (32)$$

The GPQs for σ_X^2 and σ_Y^2 are defined by:

$$R_{\sigma_X^2}=\frac{(n-1)\,s_X^2}{V_X} \text{ and } R_{\sigma_Y^2}=\frac{(m-1)\,s_Y^2}{V_Y}. \quad (33)$$

Moreover, the means are given by:

$$\mu_X\approx\bar{x}-\frac{Z_X}{\sqrt{U_X}}\sqrt{\frac{(n-1)\,s_X^2}{n}} \text{ and } \mu_Y\approx\bar{y}-\frac{Z_Y}{\sqrt{U_Y}}\sqrt{\frac{(m-1)\,s_Y^2}{m}}. \quad (34)$$

Thus, the GPQs for μ_X and μ_Y are defined by:

$$R_{\mu_X}=\bar{x}-\frac{Z_X}{\sqrt{U_X}}\sqrt{\frac{(n-1)\,s_X^2}{n}} \text{ and } R_{\mu_Y}=\bar{y}-\frac{Z_Y}{\sqrt{U_Y}}\sqrt{\frac{(m-1)\,s_Y^2}{m}}. \quad (35)$$

Therefore, the GPQ for δ is defined by:

$$R_\delta=R_{\theta_X}-R_{\theta_Y}=\frac{nR_{\mu_X}}{n+\left(R_{\sigma_X^2}/R_{\mu_X}^2\right)}-\frac{mR_{\mu_Y}}{m+\left(R_{\sigma_Y^2}/R_{\mu_Y}^2\right)}. \quad (36)$$

Moreover, the GPQ for δ^{*} is defined by:

$$R_{\delta^*}=R_{\theta_X^*}-R_{\theta_Y^*}=\frac{nR_{\mu_X}}{n-\left(R_{\sigma_X^2}/R_{\mu_X}^2\right)}-\frac{mR_{\mu_Y}}{m-\left(R_{\sigma_Y^2}/R_{\mu_Y}^2\right)}. \quad (37)$$

Therefore, the $100\,(1-\alpha)\,\%$ two-sided confidence intervals for the difference between normal means with unknown CVs based on the GCI approach are obtained by:

$$CI_{GCI.\delta}=\left(R_\delta\,(\alpha/2)\,,R_\delta\,(1-\alpha/2)\right) \quad (38)$$

and:

$$CI_{GCI.\delta^*}=\left(R_{\delta^*}\,(\alpha/2)\,,R_{\delta^*}\,(1-\alpha/2)\right), \quad (39)$$

where $R_\delta(\alpha)$ and $R_{\delta^*}(\alpha)$ denote the $100(\alpha)$-th percentiles of R_δ and R_{δ^*}, respectively.

Algorithm 3. For a given $\bar{x}$ and $\bar{y}$, the GCI for δ and δ^* can be computed by the following steps:

Step 1. Generate $V_X \sim \chi^2_{n-1}$ and $V_Y \sim \chi^2_{m-1}$, then compute $R_{\sigma^2_X}$ and $R_{\sigma^2_Y}$ from Equation (33).
Step 2. Generate $Z_X \sim N(0,1)$, $Z_Y \sim N(0,1)$, $U_X \sim \chi^2_{n-1}$ and $U_Y \sim \chi^2_{m-1}$, then compute R_{μ_X} and R_{μ_Y} from Equation (35).
Step 3. Compute R_δ from Equation (36), and compute R_{δ^*} from Equation (37).
Step 4. Repeat Steps 1–3, a total q times, and obtain an array of R_δ's and R_{δ^*}'s.
Step 5. Compute $R_\delta(\alpha/2)$, $R_\delta(1-\alpha/2)$, $R_{\delta^*}(\alpha/2)$ and $R_{\delta^*}(1-\alpha/2)$.

3.2. Large Sample Confidence Intervals for the Difference between Means of Normal Distributions with Unknown Coefficients of Variation

Again, the estimators of the difference between means with unknown CVs are defined by:

$$\hat{\delta} = \hat{\theta}_X - \hat{\theta}_Y = \frac{n\bar{X}}{n + (S^2_X/\bar{X}^2)} - \frac{m\bar{Y}}{m + (S^2_Y/\bar{Y}^2)} \tag{40}$$

and:

$$\hat{\delta}^* = \hat{\theta}^*_X - \hat{\theta}^*_Y = \frac{n\bar{X}}{n - (S^2_X/\bar{X}^2)} - \frac{m\bar{Y}}{m - (S^2_Y/\bar{Y}^2)}. \tag{41}$$

From Theorem 3, the variance of $\hat{\delta}$ is defined by:

$$Var(\hat{\delta}) = \left(\frac{\mu_X}{1+\left(\frac{\sigma^2_X}{n\mu^2_X+\sigma^2_X}\right)\left(1+\frac{2\sigma^4_X+4n\mu^2_X\sigma^2_X}{(n\mu^2_X+\sigma^2_X)^2}\right)}\right)^2 \left(\frac{\sigma^2_X}{n\mu^2_X} + \frac{\left(\frac{n\sigma^2_X}{n\mu^2_X+\sigma^2_X}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4_X+4n\mu^2_X\sigma^2_X}{(n\mu^2_X+\sigma^2_X)^2}\right)}{\left(n+\left(\frac{n\sigma^2_X}{n\mu^2_X+\sigma^2_X}\right)\left(1+\frac{2\sigma^4_X+4n\mu^2_X\sigma^2_X}{(n\mu^2_X+\sigma^2_X)^2}\right)\right)^2}\right)$$
$$+ \left(\frac{\mu_Y}{1+\left(\frac{\sigma^2_Y}{m\mu^2_Y+\sigma^2_Y}\right)\left(1+\frac{2\sigma^4_Y+4m\mu^2_Y\sigma^2_Y}{(m\mu^2_Y+\sigma^2_Y)^2}\right)}\right)^2 \left(\frac{\sigma^2_Y}{m\mu^2_Y} + \frac{\left(\frac{m\sigma^2_Y}{m\mu^2_Y+\sigma^2_Y}\right)^2\left(\frac{2}{m}+\frac{2\sigma^4_Y+4m\mu^2_Y\sigma^2_Y}{(m\mu^2_Y+\sigma^2_Y)^2}\right)}{\left(m+\left(\frac{m\sigma^2_Y}{m\mu^2_Y+\sigma^2_Y}\right)\left(1+\frac{2\sigma^4_Y+4m\mu^2_Y\sigma^2_Y}{(m\mu^2_Y+\sigma^2_Y)^2}\right)\right)^2}\right), \tag{42}$$

with μ_X, μ_Y, σ^2_X and σ^2_Y replaced by $\bar{x}$, $\bar{y}$, s^2_X and s^2_Y, respectively.

From Theorem 4, the variance of $\hat{\delta}^*$ is defined by:

$$Var(\hat{\delta}^*) = \left(\frac{\mu_X}{1-\left(\frac{\sigma^2_X}{n\mu^2_X+\sigma^2_X}\right)\left(1+\frac{2\sigma^4_X+4n\mu^2_X\sigma^2_X}{(n\mu^2_X+\sigma^2_X)^2}\right)}\right)^2 \left(\frac{\sigma^2_X}{n\mu^2_X} + \frac{\left(\frac{n\sigma^2_X}{n\mu^2_X+\sigma^2_X}\right)^2\left(\frac{2}{n}+\frac{2\sigma^4_X+4n\mu^2_X\sigma^2_X}{(n\mu^2_X+\sigma^2_X)^2}\right)}{\left(n-\left(\frac{n\sigma^2_X}{n\mu^2_X+\sigma^2_X}\right)\left(1+\frac{2\sigma^4_X+4n\mu^2_X\sigma^2_X}{(n\mu^2_X+\sigma^2_X)^2}\right)\right)^2}\right)$$
$$+ \left(\frac{\mu_Y}{1-\left(\frac{\sigma^2_Y}{m\mu^2_Y+\sigma^2_Y}\right)\left(1+\frac{2\sigma^4_Y+4m\mu^2_Y\sigma^2_Y}{(m\mu^2_Y+\sigma^2_Y)^2}\right)}\right)^2 \left(\frac{\sigma^2_Y}{m\mu^2_Y} + \frac{\left(\frac{m\sigma^2_Y}{m\mu^2_Y+\sigma^2_Y}\right)^2\left(\frac{2}{m}+\frac{2\sigma^4_Y+4m\mu^2_Y\sigma^2_Y}{(m\mu^2_Y+\sigma^2_Y)^2}\right)}{\left(m-\left(\frac{m\sigma^2_Y}{m\mu^2_Y+\sigma^2_Y}\right)\left(1+\frac{2\sigma^4_Y+4m\mu^2_Y\sigma^2_Y}{(m\mu^2_Y+\sigma^2_Y)^2}\right)\right)^2}\right), \tag{43}$$

with μ_X, μ_Y, σ^2_X and σ^2_Y replaced by $\bar{x}$, $\bar{y}$, s^2_X and s^2_Y, respectively.

Therefore, the $100(1-\alpha)\%$ two-sided confidence intervals for the difference between normal means with unknown CVs based on the LS approach are obtained by:

$$CI_{LS.\delta} = \left(\hat{\delta} - z_{1-\alpha/2}\sqrt{Var(\hat{\delta})}, \hat{\delta} + z_{1-\alpha/2}\sqrt{Var(\hat{\delta})}\right) \tag{44}$$

and:
$$CI_{LS.\delta^*} = \left(\hat{\delta^*} - z_{1-\alpha/2}\sqrt{Var(\hat{\delta^*})}, \hat{\delta^*} + z_{1-\alpha/2}\sqrt{Var(\hat{\delta^*})}\right), \tag{45}$$

where $z_{1-\alpha/2}$ denotes the $1-\alpha/2$-th quantile of the standard normal distribution.

3.3. Method of Variance Estimates Recovery Confidence Intervals for the Difference between Means of Normal Distributions with Unknown Coefficients of Variation

Since the difference between means is denoted by $\delta = \theta_X - \theta_Y$, where θ_X and θ_Y are the means of $X = (X_1, X_2, \ldots, X_n)$ and $Y = (Y_1, Y_2, \ldots, Y_m)$, respectively, suppose that $\hat{\theta}_X$ and $\hat{\theta}_Y$ are estimators of θ_X and θ_Y, respectively. The confidence intervals for θ_X and θ_Y are defined by:

$$(l_X, u_X) = \left(\hat{\theta}_X - t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_X)}, \hat{\theta}_X + t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_X)}\right) \tag{46}$$

and:
$$(l_Y, u_Y) = \left(\hat{\theta}_Y - t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_Y)}, \hat{\theta}_Y + t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_Y)}\right). \tag{47}$$

Similarly, the difference between means is denoted by $\delta^* = \theta_X^* - \theta_Y^*$. The confidence intervals for θ_X^* and θ_Y^* are defined by:

$$(l_X^*, u_X^*) = \left(\hat{\theta}_X^* - t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_X^*)}, \hat{\theta}_X^* + t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_X^*)}\right) \tag{48}$$

and:
$$(l_Y^*, u_Y^*) = \left(\hat{\theta}_Y^* - t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_Y^*)}, \hat{\theta}_Y^* + t_{1-\alpha/2}\sqrt{Var(\hat{\theta}_Y^*)}\right). \tag{49}$$

The MOVER approach, introduced by Donner and Zou [23], is used to construct the $100\,(1-\alpha)\,\%$ two-sided confidence interval (L_δ, U_δ) of $\theta_X - \theta_Y$ where L_δ and U_δ denote the lower limit and upper limit of the confidence interval, respectively. The lower limit and upper limit for δ are given by:

$$L_\delta = \hat{\theta}_X - \hat{\theta}_Y - \sqrt{\left(\hat{\theta}_X - l_X\right)^2 + \left(u_Y - \hat{\theta}_Y\right)^2} \tag{50}$$

and:
$$U_\delta = \hat{\theta}_X - \hat{\theta}_Y + \sqrt{\left(u_X - \hat{\theta}_X\right)^2 + \left(\hat{\theta}_Y - l_Y\right)^2}. \tag{51}$$

Similarly, the lower limit and upper limit for δ^* are given by:

$$L_{\delta^*} = \hat{\theta}_X^* - \hat{\theta}_Y^* - \sqrt{\left(\hat{\theta}_X^* - l_X^*\right)^2 + \left(u_Y^* - \hat{\theta}_Y^*\right)^2} \tag{52}$$

and:
$$U_{\delta^*} = \hat{\theta}_X^* - \hat{\theta}_Y^* + \sqrt{\left(u_X^* - \hat{\theta}_X^*\right)^2 + \left(\hat{\theta}_Y^* - l_Y^*\right)^2}. \tag{53}$$

Therefore, the $100\,(1-\alpha)\,\%$ two-sided confidence intervals for the difference between normal means with unknown CVs based on the MOVER approach are obtained by:

$$CI_{MOVER.\delta} = \left(\hat{\theta}_X - \hat{\theta}_Y - \sqrt{\left(\hat{\theta}_X - l_X\right)^2 + \left(u_Y - \hat{\theta}_Y\right)^2}, \hat{\theta}_X - \hat{\theta}_Y + \sqrt{\left(u_X - \hat{\theta}_X\right)^2 + \left(\hat{\theta}_Y - l_Y\right)^2}\right) \tag{54}$$

and:

$$CI_{MOVER.\delta^*} = \left(\hat{\theta}_X^* - \hat{\theta}_Y^* - \sqrt{\left(\hat{\theta}_X^* - l_X^*\right)^2 + \left(u_Y^* - \hat{\theta}_Y^*\right)^2}, \hat{\theta}_X^* - \hat{\theta}_Y^* + \sqrt{\left(u_X^* - \hat{\theta}_X^*\right)^2 + \left(\hat{\theta}_Y^* - l_Y^*\right)^2}\right). \tag{55}$$

Algorithm 4. The coverage probability for δ and δ^* can be computed by the following steps:

Step 1. Generate $X_1, X_2, \ldots, X_n$ from $N\left(\mu_X, \sigma_X^2\right)$, and then, compute $\bar{x}$ and s_X^2. Additionally, generate $Y_1, Y_2, \ldots, Y_m$ from $N\left(\mu_Y, \sigma_Y^2\right)$, and then, compute $\bar{y}$ and s_Y^2.
Step 2. Use Algorithm 3 to construct $CI_{GCI.\delta}$, and record whether or not the values of δ fall in the corresponding confidence interval.
Step 3. Use Algorithm 3 to construct $CI_{GCI.\delta^*}$, and record whether or not the values of δ^* fall in the corresponding confidence interval.
Step 4. Use Equation (44) to construct $CI_{LS.\delta}$, and record whether or not the values of δ fall in the corresponding confidence interval.
Step 5. Use Equation (45) to construct $CI_{LS.\delta^*}$, and record whether or not the values of δ^* fall in the corresponding confidence interval.
Step 6. Use Equation (54) to construct $CI_{MOVER.\delta}$, and record whether or not the values of δ fall in the corresponding confidence interval.
Step 7. Use Equation (55) to construct $CI_{MOVER.\delta^*}$, and record whether or not the values of δ^* fall in the corresponding confidence interval.
Step 8. Repeat Steps 1–7, a total M times. Then, for $CI_{GCI.\delta}$, $CI_{LS.\delta}$ and $CI_{MOVER.\delta}$, the fraction of times that all δ are in their corresponding confidence intervals provides an estimate of the coverage probability. Similarly, for $CI_{GCI.\delta^*}$, $CI_{LS.\delta^*}$ and $CI_{MOVER.\delta^*}$, the fraction of times that all δ^* are in their corresponding confidence intervals provides an estimate of the coverage probability.

4. Simulation Studies

To compare the performance of the confidence intervals, coverage probabilities and average lengths, introduced in Sections 2 and 3, two simulation studies were conducted. Comparison studies were also conducted using the Student's t-distribution, the z-distribution and the WS approach. The Student's t-distribution was used to construct the confidence interval for the single mean of the normal distribution when the sample size is small, whereas the z-distribution was used to construct the confidence interval when the sample size is large. The WS approach was used for constructing the confidence interval for the difference of the means of the normal distribution; see the paper by Niwitpong and Niwitpong [24]. The nominal confidence level of $1 - \alpha = 0.95$ was set. The confidence interval, with the values of the coverage probability greater than or close to the nominal confidence level and also having the shortest average length, was chosen.

Firstly, the performances of the confidence intervals for the single mean of the normal distribution with unknown CV (θ and θ^*) were compared. The confidence intervals were constructed with the GCI approach ($CI_{GCI.\theta}$ and $CI_{GCI.\theta^*}$) and the LS approach ($CI_{LS.\theta}$ and $CI_{LS.\theta^*}$). Furthermore, the standard confidence interval for the single mean of the normal distribution (CI_μ) was constructed based on the Student's t-distribution and the z-distribution. Algorithm 1 and Algorithm 2 were used to compute coverage probabilities and average lengths with $q = 2500$ and $M = 5000$ of sample size n from $N(\mu, \sigma^2)$ for $\mu = 1.0$, $\sigma = 0.3$, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0 and $n = 10$, 20, 30, 50, 100. The CVs were computed by σ/μ. Tables 1 and 2 show the coverage probabilities and average lengths of the 95% two-sided confidence intervals for θ, θ^* and μ. The results indicated that the GCIs are similar to the paper by Sodanin et al. [16] in terms of coverage probability and average length. For the GCI approach, $CI_{GCI.\theta}$ provides better confidence interval estimates than $CI_{GCI.\theta^*}$ in almost all cases. This is because the coverage probabilities of $CI_{GCI.\theta^*}$ are close to 1.00 when σ increases. Hence, $CI_{GCI.\theta^*}$ is a conservative confidence interval when σ increases. For the LS approach, the coverage probabilities of $CI_{LS.\theta}$ and $CI_{LS.\theta^*}$ provide less than the nominal confidence level of 0.95 and are close to 1.00 when σ increases. Therefore, the LS approach is not recommended to construct the confidence interval for the single mean of the normal distribution with unknown CV. This is then compared with CI_μ. For a small sample size, the coverage probability of $CI_{GCI.\theta}$ performs as well as that of CI_μ. The length of CI_μ is a bit shorter than the length of $CI_{GCI.\theta}$. Hence, CI_μ is better than $CI_{GCI.\theta}$ in terms of the average length when the sample size is small. For a large sample size, $CI_{GCI.\theta}$ is better than CI_μ in terms of coverage probability. Furthermore, the coverage probability of $CI_{GCI.\theta}$ is more stable than that of CI_{μ} in all sample size cases.

Table 1. The coverage probabilities of 95% of the two-sided confidence intervals for the mean of the normal distribution with the unknown coefficient of variation (CV).

n	σ	θ		θ^*		CI_{μ}
		$CI_{GCI.\theta}$	$CI_{LS.\theta}$	$CI_{GCI.\theta^*}$	$CI_{LS.\theta^*}$	
10	0.3	0.9472	0.9126	0.9492	0.9204	0.9496
	0.5	0.9486	0.9056	0.9546	0.9326	0.9500
	0.7	0.9524	0.9060	0.9616	0.9526	0.9510
	0.9	0.9528	0.8882	0.9726	0.9748	0.9518
	1.0	0.9480	0.8802	0.9754	0.9686	0.9486
	1.1	0.9512	0.8754	0.9768	0.9660	0.9496
	1.3	0.9468	0.8464	0.9742	0.9248	0.9468
	1.5	0.9474	0.8348	0.9796	0.8728	0.9482
	1.7	0.9534	0.8142	0.9880	0.8166	0.9526
	2.0	0.9536	0.7904	0.9956	0.7364	0.9540
20	0.3	0.9548	0.9408	0.9538	0.9430	0.9556
	0.5	0.9456	0.9268	0.9464	0.9412	0.9462
	0.7	0.9460	0.9264	0.9470	0.9504	0.9460
	0.9	0.9488	0.9296	0.9576	0.9702	0.9480
	1.0	0.9492	0.9330	0.9650	0.9824	0.9478
	1.1	0.9510	0.9284	0.9688	0.9842	0.9506
	1.3	0.9478	0.9178	0.9728	0.9858	0.9508
	1.5	0.9524	0.9052	0.9760	0.9730	0.9522
	1.7	0.9490	0.8834	0.9708	0.9460	0.9494
	2.0	0.9524	0.8446	0.9752	0.8870	0.9524
30	0.3	0.9546	0.9424	0.9538	0.9432	0.9430
	0.5	0.9502	0.9360	0.9496	0.9440	0.9390
	0.7	0.9512	0.9382	0.9508	0.9520	0.9416
	0.9	0.9494	0.9402	0.9518	0.9672	0.9414
	1.0	0.9512	0.9464	0.9594	0.9778	0.9428
	1.1	0.9488	0.9452	0.9616	0.9852	0.9370
	1.3	0.9498	0.9482	0.9714	0.9968	0.9412
	1.5	0.9508	0.9420	0.9732	0.9938	0.9406
	1.7	0.9508	0.9194	0.9738	0.9840	0.9432
	2.0	0.9456	0.8804	0.9710	0.9554	0.9342
50	0.3	0.9506	0.9456	0.9508	0.9474	0.9466
	0.5	0.9490	0.9432	0.9496	0.9466	0.9450
	0.7	0.9474	0.9378	0.9480	0.9492	0.9406
	0.9	0.9530	0.9480	0.9502	0.9642	0.9474
	1.0	0.9442	0.9444	0.9460	0.9700	0.9384
	1.1	0.9488	0.9532	0.9496	0.9774	0.9428
	1.3	0.9460	0.9620	0.9534	0.9930	0.9414
	1.5	0.9482	0.9656	0.9642	0.9978	0.9436
	1.7	0.9508	0.9564	0.9736	0.9984	0.9434
	2.0	0.9502	0.9338	0.9778	0.9944	0.9446
100	0.3	0.9520	0.9490	0.9522	0.9504	0.9496
	0.5	0.9470	0.9436	0.9472	0.9460	0.9446
	0.7	0.9468	0.9410	0.9462	0.9478	0.9434
	0.9	0.9484	0.9474	0.9482	0.9558	0.9480
	1.0	0.9540	0.9546	0.9538	0.9648	0.9504
	1.1	0.9482	0.9526	0.9472	0.9656	0.9460
	1.3	0.9504	0.9658	0.9522	0.9846	0.9494
	1.5	0.9484	0.9748	0.9492	0.9960	0.9468
	1.7	0.9512	0.9790	0.9516	0.9994	0.9482
	2.0	0.9512	0.9724	0.9638	1.0000	0.9484

Table 2. The average lengths of 95% of two-sided confidence intervals for the mean of the normal distribution with unknown CV.

n	σ	θ		θ^*		CI_μ
		$CI_{GCI.\theta}$	$CI_{LS.\theta}$	$CI_{GCI.\theta^*}$	$CI_{LS.\theta^*}$	
10	0.3	0.4253	0.3608	0.4137	0.3687	0.4186
	0.5	0.7245	0.6007	0.6744	0.6490	0.6963
	0.7	1.0259	0.8642	1.1274	1.5207	0.9762
	0.9	1.2803	1.1520	2.0372	11.1622	1.2454
	1.0	1.3992	1.3011	2.7009	27.4894	1.3893
	1.1	1.5259	1.4717	3.4517	64.8564	1.5438
	1.3	1.7277	1.7115	4.9087	43.8045	1.8119
	1.5	1.9375	1.9505	6.2964	110.1241	2.0860
	1.7	2.1402	2.0783	7.7439	45.9743	2.3560
	2.0	2.4683	2.3036	9.8795	22.8761	2.7808
20	0.3	0.2780	0.2580	0.2751	0.2605	0.2765
	0.5	0.4709	0.4313	0.4556	0.4443	0.4632
	0.7	0.6700	0.6141	0.6230	0.6627	0.6489
	0.9	0.8737	0.8322	0.8079	1.3335	0.8338
	1.0	0.9713	0.9546	0.9411	56.2417	0.9240
	1.1	1.0682	1.0867	1.1468	24.2131	1.0175
	1.3	1.2466	1.3373	1.7581	20.3697	1.2060
	1.5	1.4052	1.5424	2.5559	20.0763	1.3916
	1.7	1.5542	1.6978	3.4312	22.5032	1.5748
	2.0	1.7605	1.8027	4.9410	19.9994	1.8496
30	0.3	0.2233	0.2125	0.2218	0.2139	0.2130
	0.5	0.3743	0.3535	0.3670	0.3600	0.3551
	0.7	0.5295	0.4998	0.5077	0.5211	0.4973
	0.9	0.6859	0.6659	0.6339	0.7431	0.6361
	1.0	0.7692	0.7680	0.6993	0.9611	0.7089
	1.1	0.8508	0.8789	0.7764	2.0789	0.7795
	1.3	1.0151	1.1182	1.0086	11.3310	0.9249
	1.5	1.1552	1.3209	1.3581	26.2084	1.0586
	1.7	1.2986	1.4784	1.9502	21.8649	1.2103
	2.0	1.4842	1.6205	2.9085	14.7054	1.4206
50	0.3	0.1706	0.1657	0.1699	0.1663	0.1659
	0.5	0.2842	0.2746	0.2811	0.2776	0.2755
	0.7	0.3996	0.3859	0.3907	0.3947	0.3853
	0.9	0.5201	0.5127	0.4998	0.5381	0.4978
	1.0	0.5783	0.5823	0.5491	0.6310	0.5509
	1.1	0.6397	0.6651	0.5992	0.7966	0.6067
	1.3	0.7639	0.8559	0.6957	2.0416	0.7173
	1.5	0.8875	1.0531	0.8086	24.8041	0.8267
	1.7	1.0090	1.2198	0.9864	40.4766	0.9368
	2.0	1.1799	1.3726	1.4258	15.3103	1.1031
100	0.3	0.1188	0.1172	0.1186	0.1174	0.1173
	0.5	0.1981	0.1948	0.1971	0.1958	0.1951
	0.7	0.2791	0.2742	0.2762	0.2771	0.2741
	0.9	0.3595	0.3569	0.3532	0.3638	0.3519
	1.0	0.3997	0.4024	0.3909	0.4137	0.3907
	1.1	0.4410	0.4534	0.4291	0.4735	0.4299
	1.3	0.5238	0.5745	0.5032	0.6547	0.5078
	1.5	0.6089	0.7220	0.5747	1.4514	0.5864
	1.7	0.6960	0.8757	0.6435	29.4149	0.6654
	2.0	0.8270	1.0523	0.7472	26.8938	0.7831

The second simulation study was to compare the performance of confidence intervals for the difference between two means of normal distributions with unknown CVs (δ and δ^*). There are three approaches; GCIs are defined as $CI_{GCI.\delta}$ and $CI_{GCI.\delta^*}$; large sample confidence intervals are defined as $CI_{LS.\delta}$ and $CI_{LS.\delta^*}$; and MOVER confidence intervals are defined as $CI_{MOVER.\delta}$ and $CI_{MOVER.\delta^*}$ compared with the WS confidence interval for the difference of the means of the normal distribution ($CI_{\mu_X-\mu_Y}$). Algorithm 3 and Algorithm 4 were used to compute coverage probabilities and average lengths with $q = 2500$ and $M = 5000$. The sample sizes n from $N(\mu_X, \sigma_X^2)$ and m from $N(\mu_Y, \sigma_Y^2)$ for the sample sizes were (n, m) = (10,10), (10,20), (30,30), (20,30), (50,50), (30,50), (100,100) and (50,100). The population means were $\mu_X = \mu_Y = \mu = 1.0$, and the population standard deviations were $\sigma_X = 0.3$, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 1.7, 2.0 and $\sigma_Y = 1.0$. The coefficients of variation were computed by $\tau_X = \sigma_X/\mu_X$ and $\tau_Y = \sigma_Y/\mu_Y$; also, the ratio of τ_X to τ_Y reduces to σ_X/σ_Y when we set $\mu_X = \mu_Y$. Tables 3 and 4 show that the coverage probabilities and average lengths of 95% two-sided confidence intervals for δ, δ^* and $\mu_X - \mu_Y$. For the GCI approach, the coverage probabilities of $CI_{GCI.\delta}$ are close to the nominal confidence level of 0.95 for all cases. For small sample sizes, $CI_{GCI.\delta^*}$ is the conservative confidence interval because the coverage probabilities are in the range from 0.97–1.00. Moreover, the coverage probabilities of $CI_{GCI.\delta^*}$ are close to the nominal confidence level of 0.95 when the sample sizes (n and m) increase. For the LS approach, $CI_{LS.\delta}$ have the coverage probabilities under the nominal confidence level of 0.95 and close to the nominal confidence level of 0.95 when the sample sizes are large. Furthermore, $CI_{LS.\delta^*}$ is a conservative confidence interval because the coverage probabilities are close to 1.00. For the MOVER approach, the coverage probability of $CI_{MOVER.\delta}$ is not stable, whereas $CI_{MOVER.\delta^*}$ is a conservative confidence interval. In addition, $CI_{GCI.\delta}$ is better than $CI_{\mu_X-\mu_Y}$ in terms of coverage probability.

Table 3. The coverage probabilities of 95% of two-sided confidence intervals for the difference between the means of the normal distributions with unknown CVs.

n	m	$\frac{\sigma_X}{\sigma_Y}$	δ			δ^*			$CI_{\mu_X-\mu_Y}$
			$CI_{GCI.\delta}$	$CI_{LS.\delta}$	$CI_{MOVER.\delta}$	$CI_{GCI.\delta^*}$	$CI_{LS.\delta^*}$	$CI_{MOVER.\delta^*}$	
10	10	0.3	0.9534	0.8890	0.9170	0.9744	0.9744	0.9838	0.9472
		0.5	0.9568	0.8980	0.9262	0.9762	0.9758	0.9864	0.9474
		0.7	0.9638	0.9158	0.9450	0.9840	0.9846	0.9906	0.9536
		0.9	0.9646	0.9240	0.9496	0.9900	0.9844	0.9902	0.9524
		1.0	0.9636	0.9206	0.9494	0.9928	0.9846	0.9908	0.9512
		1.1	0.9624	0.9154	0.9468	0.9942	0.9796	0.9868	0.9488
		1.3	0.9642	0.9186	0.9444	0.9968	0.9630	0.9750	0.9504
		1.5	0.9604	0.9166	0.9454	0.9948	0.9374	0.9570	0.9506
		1.7	0.9606	0.9138	0.9424	0.9970	0.9088	0.9360	0.9516
		2.0	0.9624	0.9202	0.9466	0.9984	0.8472	0.8832	0.9508
10	20	0.3	0.9548	0.9370	0.9508	0.9672	0.9810	0.9878	0.9500
		0.5	0.9582	0.9328	0.9520	0.9708	0.9794	0.9880	0.9500
		0.7	0.9610	0.9298	0.9536	0.9702	0.9800	0.9898	0.9504
		0.9	0.9562	0.9236	0.9466	0.9798	0.9866	0.9914	0.9470
		1.0	0.9594	0.9204	0.9444	0.9838	0.9846	0.9910	0.9512
		1.1	0.9618	0.9162	0.9362	0.9874	0.9772	0.9846	0.9548
		1.3	0.9562	0.8992	0.9240	0.9856	0.9492	0.9676	0.9484
		1.5	0.9556	0.8868	0.9086	0.9870	0.9056	0.9328	0.9462
		1.7	0.9524	0.8810	0.9076	0.9892	0.8568	0.8966	0.9438
		2.0	0.9560	0.8778	0.9074	0.9946	0.7704	0.8050	0.9506

Table 3. *Cont.*

n	m	$\frac{\sigma_X}{\sigma_Y}$	δ			δ^*			$CI_{\mu_X-\mu_Y}$
			$CI_{GCI.\delta}$	$CI_{LS.\delta}$	$CI_{MOVER.\delta}$	$CI_{GCI.\delta^*}$	$CI_{LS.\delta^*}$	$CI_{MOVER.\delta^*}$	
30	30	0.3	0.9524	0.9458	0.9552	0.9566	0.9804	0.9834	0.9528
		0.5	0.9492	0.9400	0.9492	0.9548	0.9744	0.9806	0.9450
		0.7	0.9530	0.9448	0.9544	0.9544	0.9756	0.9814	0.9492
		0.9	0.9512	0.9474	0.9566	0.9548	0.9774	0.9818	0.9470
		1.0	0.9560	0.9564	0.9630	0.9614	0.9860	0.9892	0.9514
		1.1	0.9520	0.9508	0.9578	0.9604	0.9886	0.9910	0.9486
		1.3	0.9542	0.9578	0.9646	0.9680	0.9946	0.9960	0.9498
		1.5	0.9532	0.9550	0.9606	0.9714	0.9934	0.9940	0.9504
		1.7	0.9496	0.9362	0.9398	0.9734	0.9852	0.9858	0.9472
		2.0	0.9518	0.9176	0.9238	0.9726	0.9580	0.9584	0.9494
20	30	0.3	0.9502	0.9430	0.9514	0.9542	0.9758	0.9806	0.9472
		0.5	0.9520	0.9438	0.9560	0.9548	0.9736	0.9794	0.9476
		0.7	0.9486	0.9374	0.9488	0.9530	0.9766	0.9814	0.9434
		0.9	0.9534	0.9452	0.9562	0.9622	0.9828	0.9866	0.9490
		1.0	0.9566	0.9494	0.9578	0.9674	0.9894	0.9920	0.9502
		1.1	0.9536	0.9422	0.9528	0.9700	0.9914	0.9938	0.9490
		1.3	0.9556	0.9410	0.9494	0.9732	0.9904	0.9916	0.9512
		1.5	0.9504	0.9266	0.9362	0.9766	0.9788	0.9796	0.9458
		1.7	0.9558	0.9064	0.9160	0.9768	0.9498	0.9522	0.9534
		2.0	0.9496	0.8770	0.8848	0.9792	0.8968	0.9010	0.9460
50	50	0.3	0.9490	0.9474	0.9512	0.9486	0.9658	0.9686	0.9472
		0.5	0.9470	0.9480	0.9528	0.9494	0.9664	0.9700	0.9460
		0.7	0.9510	0.9480	0.9526	0.9520	0.9652	0.9700	0.9480
		0.9	0.9520	0.9514	0.9600	0.9522	0.9712	0.9756	0.9498
		1.0	0.9516	0.9522	0.9586	0.9510	0.9750	0.9780	0.9504
		1.1	0.9516	0.9566	0.9616	0.9524	0.9800	0.9828	0.9510
		1.3	0.9510	0.9656	0.9684	0.9532	0.9940	0.9952	0.9480
		1.5	0.9528	0.9670	0.9708	0.9666	0.9986	0.9990	0.9520
		1.7	0.9510	0.9636	0.9678	0.9722	0.9990	0.9990	0.9514
		2.0	0.9478	0.9402	0.9444	0.9716	0.9922	0.9922	0.9482
30	50	0.3	0.9492	0.9466	0.9524	0.9484	0.9664	0.9710	0.9466
		0.5	0.9530	0.9510	0.9572	0.9530	0.9680	0.9714	0.9504
		0.7	0.9482	0.9468	0.9522	0.9492	0.9646	0.9716	0.9460
		0.9	0.9548	0.9518	0.9590	0.9558	0.9760	0.9802	0.9524
		1.0	0.9544	0.9530	0.9598	0.9550	0.9802	0.9840	0.9510
		1.1	0.9470	0.9486	0.9564	0.9554	0.9844	0.9870	0.9444
		1.3	0.9536	0.9554	0.9610	0.9690	0.9960	0.9966	0.9508
		1.5	0.9548	0.9480	0.9530	0.9732	0.9938	0.9942	0.9496
		1.7	0.9498	0.9318	0.9374	0.9738	0.9846	0.9850	0.9490
		2.0	0.9534	0.9056	0.9132	0.9746	0.9568	0.9568	0.9520
100	100	0.3	0.9484	0.9510	0.9536	0.9492	0.9626	0.9642	0.9484
		0.5	0.9514	0.9510	0.9524	0.9510	0.9594	0.9624	0.9498
		0.7	0.9512	0.9508	0.9536	0.9508	0.9584	0.9606	0.9504
		0.9	0.9480	0.9496	0.9528	0.9480	0.9586	0.9608	0.9480
		1.0	0.9494	0.9524	0.9544	0.9496	0.9614	0.9648	0.9504
		1.1	0.9528	0.9558	0.9580	0.9538	0.9670	0.9674	0.9528
		1.3	0.9482	0.9616	0.9642	0.9484	0.9782	0.9798	0.9456
		1.5	0.9576	0.9774	0.9782	0.9584	0.9964	0.9968	0.9570
		1.7	0.9542	0.9784	0.9802	0.9546	0.9994	0.9994	0.9536
		2.0	0.9516	0.9762	0.9772	0.9602	0.9998	0.9998	0.9504
50	100	0.3	0.9504	0.9516	0.9544	0.9506	0.9590	0.9618	0.9512
		0.5	0.9516	0.9498	0.9546	0.9516	0.9586	0.9614	0.9508
		0.7	0.9596	0.9598	0.9624	0.9604	0.9662	0.9688	0.9600
		0.9	0.9576	0.9560	0.9614	0.9590	0.9686	0.9718	0.9572
		1.0	0.9496	0.9510	0.9542	0.9486	0.9690	0.9728	0.9488
		1.1	0.9532	0.9570	0.9610	0.9550	0.9760	0.9798	0.9518
		1.3	0.9508	0.9634	0.9676	0.9550	0.9924	0.9930	0.9514
		1.5	0.9526	0.9672	0.9696	0.9644	0.9974	0.9982	0.9512
		1.7	0.9492	0.9586	0.9614	0.9694	0.9996	0.9996	0.9486
		2.0	0.9478	0.9352	0.9384	0.9710	0.9946	0.9946	0.9456

Table 4. The average lengths of 95% of two-sided confidence intervals for the difference between the means of the normal distributions with unknown CVs.

n	m	$\frac{\sigma_X}{\sigma_Y}$	δ			δ^*			$CI_{\mu_X-\mu_Y}$
			$CI_{GCI.\delta}$	$CI_{LS.\delta}$	$CI_{MOVER.\delta}$	$CI_{GCI.\delta^*}$	$CI_{LS.\delta^*}$	$CI_{MOVER.\delta^*}$	
10	10	0.3	1.4718	1.3608	1.5706	2.7495	174.6926	201.6273	1.4236
		0.5	1.5914	1.4527	1.6767	2.8520	174.7595	201.7044	1.4971
		0.7	1.7571	1.6056	1.8532	3.1970	33.8930	39.1187	1.6166
		0.9	1.9389	1.8005	2.0781	4.1762	26.1286	30.1572	1.7813
		1.0	2.0215	1.9182	2.2139	4.8029	31.8030	36.7065	1.8667
		1.1	2.0976	2.0093	2.3191	5.3971	42.2349	48.7468	1.9499
		1.3	2.2818	2.2054	2.5454	6.9736	53.7877	62.0809	2.1712
		1.5	2.4646	2.4389	2.8149	8.4225	45.4266	52.4306	2.3985
		1.7	2.6356	2.6138	3.0168	9.9089	122.9135	141.8647	2.6292
		2.0	2.9148	3.1079	3.5871	11.8712	69.6795	80.4229	2.9979
10	20	0.3	1.0621	1.0243	1.1061	1.0587	2.4146	2.5885	0.9930
		0.5	1.2152	1.1368	1.2436	1.2119	3.8426	4.1307	1.1157
		0.7	1.4160	1.3040	1.4448	1.5965	3.4517	3.7802	1.2893
		0.9	1.6279	1.5328	1.7167	2.4886	44.2464	50.8401	1.5007
		1.0	1.7222	1.6489	1.8530	3.0562	35.0240	40.1845	1.6063
		1.1	1.8227	1.7801	2.0077	3.7729	47.6685	54.8935	1.7281
		1.3	2.0132	2.0119	2.2794	5.1821	91.2530	105.0995	1.9692
		1.5	2.2017	2.1958	2.4938	6.7121	39.5264	45.4867	2.2244
		1.7	2.4089	2.4177	2.7518	8.0771	37.9031	43.5650	2.4918
		2.0	2.6824	2.5396	2.8924	10.1815	26.2631	30.1053	2.8703
30	30	0.3	0.8027	0.8006	0.8354	0.7367	1.0259	1.0706	0.7697
		0.5	0.8548	0.8481	0.8850	0.7937	1.0455	1.0910	0.8173
		0.7	0.9352	0.9223	0.9624	0.8702	1.1191	1.1678	0.8906
		0.9	1.0312	1.0236	1.0681	0.9503	1.2716	1.3269	0.9774
		1.0	1.0899	1.0974	1.1451	1.0040	1.6832	1.7564	1.0298
		1.1	1.1485	1.1760	1.2272	1.0627	2.1235	2.2159	1.0820
		1.3	1.2722	1.3654	1.4248	1.2585	8.2640	8.6235	1.1953
		1.5	1.3966	1.6270	1.6978	1.5903	27.1841	28.3667	1.3164
		1.7	1.5139	1.6831	1.7563	2.1176	35.3634	36.9019	1.4409
		2.0	1.6822	1.8095	1.8882	3.0681	21.9906	22.9473	1.6353
20	30	0.3	0.8187	0.8130	0.8505	0.7549	1.0180	1.0641	0.7827
		0.5	0.9041	0.8867	0.9306	0.8386	1.0790	1.1310	0.8589
		0.7	1.0223	0.9911	1.0438	0.9460	1.2008	1.2627	0.9640
		0.9	1.1661	1.1438	1.2085	1.0920	1.8674	1.9735	1.0926
		1.0	1.2419	1.2385	1.3106	1.2215	4.2819	4.5571	1.1617
		1.1	1.3146	1.3381	1.4178	1.4014	7.5387	8.0317	1.2312
		1.3	1.4642	1.5546	1.6505	1.9154	17.0820	18.2339	1.3827
		1.5	1.6109	1.7381	1.8472	2.6707	19.0229	20.3057	1.5445
		1.7	1.7539	1.8779	1.9969	3.5847	32.3002	34.4873	1.7187
		2.0	1.9472	2.0091	2.1372	4.9841	19.0434	20.3310	1.9677
50	50	0.3	0.6034	0.6069	0.6223	0.5754	0.6541	0.6707	0.5886
		0.5	0.6447	0.6461	0.6624	0.6177	0.6911	0.7086	0.6282
		0.7	0.7038	0.7017	0.7195	0.6749	0.7477	0.7666	0.6841
		0.9	0.7786	0.7791	0.7989	0.7433	0.8346	0.8557	0.7542
		1.0	0.8194	0.8281	0.8490	0.7781	0.8994	0.9221	0.7917
		1.1	0.8638	0.8895	0.9121	0.8145	1.0048	1.0302	0.8330
		1.3	0.9584	1.0388	1.0651	0.8903	3.1159	3.1947	0.9186
		1.5	1.0597	1.2056	1.2361	0.9939	12.1816	12.4899	1.0100
		1.7	1.1659	1.3568	1.3912	1.1583	17.5067	17.9499	1.1079
		2.0	1.3165	1.4928	1.5306	1.5863	15.3073	15.6948	1.2567

Table 4. *Cont.*

n	m	$\frac{\sigma_X}{\sigma_Y}$	δ			δ^*			$CI_{\mu_X-\mu_Y}$
			$CI_{GCI.\delta}$	$CI_{LS.\delta}$	$CI_{MOVER.\delta}$	$CI_{GCI.\delta^*}$	$CI_{LS.\delta^*}$	$CI_{MOVER.\delta^*}$	
30	50	0.3	0.6209	0.6224	0.6396	0.5931	0.6690	0.6873	0.6043
		0.5	0.6903	0.6847	0.7054	0.6619	0.7306	0.7524	0.6688
		0.7	0.7836	0.7695	0.7950	0.7472	0.8216	0.8484	0.7557
		0.9	0.9015	0.8939	0.9258	0.8436	0.9923	1.0280	0.8637
		1.0	0.9654	0.9729	1.0088	0.8944	1.1737	1.2182	0.9209
		1.1	1.0300	1.0608	1.1009	0.9589	2.1593	2.2476	0.9791
		1.3	1.1651	1.2625	1.3124	1.1732	32.7157	34.1355	1.1042
		1.5	1.3020	1.4566	1.5155	1.5471	17.5557	18.3175	1.2389
		1.7	1.4241	1.5969	1.6622	2.0171	35.1323	36.6591	1.3673
		2.0	1.5982	1.7241	1.7952	2.9794	11.3044	11.7944	1.5689
100	100	0.3	0.4175	0.4195	0.4247	0.4090	0.4306	0.4359	0.4125
		0.5	0.4470	0.4480	0.4535	0.4386	0.4588	0.4644	0.4412
		0.7	0.4882	0.4877	0.4937	0.4792	0.4988	0.5050	0.4812
		0.9	0.5380	0.5388	0.5455	0.5272	0.5520	0.5588	0.5298
		1.0	0.5658	0.5704	0.5775	0.5534	0.5867	0.5940	0.5569
		1.1	0.5957	0.6076	0.6151	0.5810	0.6305	0.6383	0.5855
		1.3	0.6610	0.7056	0.7143	0.6392	0.7828	0.7925	0.6473
		1.5	0.7286	0.8282	0.8385	0.6955	1.6204	1.6405	0.7106
		1.7	0.8031	0.9658	0.9778	0.7534	36.5017	36.9534	0.7790
		2.0	0.9168	1.1278	1.1417	0.8449	18.0070	18.2298	0.8820
50	100	0.3	0.4352	0.4361	0.4424	0.4267	0.4470	0.4533	0.4293
		0.5	0.4912	0.4886	0.4967	0.4821	0.4998	0.5080	0.4834
		0.7	0.5658	0.5595	0.5699	0.5534	0.5739	0.5845	0.5555
		0.9	0.6548	0.6514	0.6647	0.6334	0.6786	0.6925	0.6396
		1.0	0.7051	0.7120	0.7270	0.6762	0.7595	0.7757	0.6867
		1.1	0.7558	0.7801	0.7971	0.7164	0.8804	0.9001	0.7332
		1.3	0.8639	0.9477	0.9694	0.8013	2.4805	2.5415	0.8314
		1.5	0.9734	1.1261	1.1528	0.9099	9.0286	9.2562	0.9311
		1.7	1.0905	1.2906	1.3216	1.0872	36.8034	37.7344	1.0400
		2.0	1.2480	1.4371	1.4719	1.5106	11.7799	12.0774	1.1961

5. An Empirical Application

Three examples are given to illustrate our proposed approaches.

Example 1. *The dataset, previously considered by Niwitpong [9], is fitted by the normal distribution. The data shows the cholesterol level of 15 participants who were given eight weeks of training to truly reduce the cholesterol level. The $n = 15$ participants were 129, 131, 154, 172, 115, 126, 175, 191, 122, 238, 159, 156, 176, 175 and 126. The sample mean and sample variance of the data were 156.3333 and 1094.9520, respectively. The sample CV was 0.2117. The GCIs for the mean with unknown CV θ and θ^* were, respectively, (136.8439, 173.6214) and (137.9615, 174.5876) with interval lengths of 36.7775 and 36.6261. The large sample confidence intervals for the mean with unknown CV θ and θ^* were (−4679.6220, 4991.3580) and (−4709.2810, 5022.8840) with interval lengths of 9670.9800 and 9732.1650, respectively. Finally, the confidence interval for the mean μ based on the Student's t-distribution was (138.0087, 174.6580) with an interval length of 36.6493.*

The simulation results are presented in Table 5. The coverage probability of $CI_{GCI.\theta}$ is as good as the coverage probability of CI_{μ}. The length of CI_{μ} provides a bit shorter length of $CI_{GCI.\theta}$. Hence, the confidence interval based on the Student's t-distribution is better than the other confidence intervals when the sample size is small. Therefore, these results confirm the simulation results for a small sample size in the previous section.

Table 5. The coverage probability (average length) of 95% of two-sided confidence intervals for the mean of the normal distribution with unknown CV when $n = 15$, $\mu = 156.3333$ and $\sigma^2 = 1094.9520$.

θ		θ^*		CI_{μ}
$CI_{GCI.\theta}$	$CI_{LS.\theta}$	$CI_{GCI.\theta^*}$	$CI_{LS.\theta^*}$	
0.9566 (36.1795)	1.0000 (11,447.5700)	0.9548 (35.9020)	1.0000 (11,523.9500)	0.9560 (35.9821)

Example 2. *The dataset, also provided by Niwitpong [9], is fitted by the normal distribution. The data show the number of defects in 100,000 lines of code in a particular type of software program made in United States and Japan. The $n = 32$ observations were as follows 48, 54, 50, 38, 39, 48, 48, 38, 42, 52, 42, 36, 52, 55, 40, 40, 40, 43, 43, 40, 48, 46, 48, 48, 52, 48, 50, 48, 52, 52, 46 and 45. The sample mean and sample variance of the data were 45.9688 and 27.7732, respectively. The CV was 0.1146. The GCIs for the mean with unknown CV θ and θ^* were, respectively, (44.0826, 47.7486) and (44.1216, 47.7818) with interval lengths of 3.6660 and 3.6602. The large sample confidence intervals for the mean with unknown CV θ and θ^* were (−1834.1070, 1926.0070) and (−8348.0830, 8440.0580) with interval lengths of 3760.1140 and 16,788.1410, respectively. Finally, the confidence interval for the mean μ based on the z-distribution was (44.1428, 47.7947) with an interval length of 3.6519.*

The simulation results are presented in Table 6. The confidence interval based on the z-distribution yields an interval length shorter than the other confidence intervals. However, the coverage probabilities of the GCI are much closer to the nominal confidence level of 0.95 than those of other confidence intervals. Therefore, the GCI approach provides the best confidence interval when the sample size is large. Hence, the results support the simulation results for large sample size in the previous section.

Table 6. The coverage probability (average length) of 95% of two-sided confidence intervals for the mean of the normal distribution with unknown CV when $n = 32$, $\mu = 45.9688$ and $\sigma^2 = 27.7732$.

θ		θ^*		CI_{μ}
$CI_{GCI.\theta}$	$CI_{LS.\theta}$	$CI_{GCI.\theta^*}$	$CI_{LS.\theta^*}$	
0.9480 (3.7747)	1.0000 (3521.8090)	0.9480 (3.7713)	1.0000 (85,798.7900)	0.9396 (3.6227)

Example 3. *The data example is taken from Lee and Lin [25] and was originally given by Jarvis et al. [26] and Pagano and Gauvreau [27]. The data are fitted by the normal distribution, representing carboxyhemoglobin levels for nonsmokers and cigarette smokers. The summary statistics of nonsmokers were $n = 121$, $\bar{x} = 1.3000$ and $s_X^2 = 1.7040$. For cigarette smokers, the summary statistics were $m = 75$, $\bar{y} = 4.1000$, and $s_Y^2 = 4.0540$. The CVs of nonsmoker and cigarette smoker were 1.0041 and 0.4911, respectively. The difference between $\bar{x}$ and $\bar{y}$ was −2.8000. The GCIs for the difference between two means with unknown CVs δ and δ^* were, respectively, (−3.3269, −2.2880) and (−3.3260, −2.2956) with interval lengths of 1.0389 and 1.0304. The large sample confidence intervals for the difference between two means with unknown CVs δ and δ^* were, respectively, (−5.2288, −0.3664) and (−5.8213, 0.2167) with interval lengths of 4.8624 and 6.0380. The MOVER confidence intervals for the difference between two means with unknown CVs δ and δ^* were, respectively, (−5.2690, −0.3262) and (−5.8714, 0.2668) with interval lengths of 4.9428 and 6.1382. Finally, the WS confidence interval for the difference between two means $\mu_X - \mu_Y$ was (−3.3172, −2.2828) with an interval length of 1.0344.*

Table 7 presents the simulation results. The GCI approach and the WS confidence interval have yielded a minimum coverage probability at 0.95. The length of one of the GCI approach, $CI_{GCI.\delta^*}$, is a bit shorter than the length of $CI_{\mu_X-\mu_Y}$. The coverage probability of $CI_{GCI.\delta}$ is better than that of $CI_{\mu_X-\mu_Y}$. Hence, the GCI approach performs well in terms of the coverage probability. Therefore, these results confirm the simulation results in the previous section.

Table 7. The coverage probability (average length) of 95% of two-sided confidence intervals for the difference between the means of the normal distributions with unknown CVs when $n = 121$, $m = 75$, $\mu_X = 1.3000$, $\mu_Y = 4.1000$, $\sigma_X^2 = 1.7040$ and $\sigma_Y^2 = 4.0540$.

δ			δ^*			$CI_{\mu_X-\mu_Y}$
$CI_{GCI.\delta}$	$CI_{LS.\delta}$	$CI_{MOVER.\delta}$	$CI_{GCI.\delta^*}$	$CI_{LS.\delta^*}$	$CI_{MOVER.\delta^*}$	
0.9512 (1.0416)	1.0000 (4.9047)	1.0000 (4.9858)	0.9506 (1.0321)	1.0000 (6.4812)	1.0000 (6.5886)	0.9502 (1.0325)

6. Discussion and Conclusions

Sodanin et al. [16] constructed the GCIs for the mean of the normal distribution with unknown CV. This paper provides generalized confidence intervals ($CI_{GCI.\theta}$ and $CI_{GCI.\theta^*}$) and proposes large sample confidence intervals ($CI_{LS.\theta}$ and $CI_{LS.\theta^*}$) for the single mean of the normal distribution with unknown CV (θ and θ^*). Comparison studies were also conducted using the standard confidence interval for the normal mean (CI_{μ}) based on the Student's t-distribution and the z-distribution, which are much more simple and easier to implement. Moreover, the new confidence intervals were proposed for the difference between two means of the normal distributions with unknown CVs (δ and δ^*). The confidence intervals for δ and δ^* were constructed based on the GCI approach ($CI_{GCI.\delta}$ and $CI_{GCI.\delta^*}$), the LS approach ($CI_{LS.\delta}$ and $CI_{LS.\delta^*}$) and the MOVER approach ($CI_{MOVER.\delta}$ and $CI_{MOVER.\delta^*}$), compared with the standard confidence interval, using the WS approach to construct the confidence interval for the difference of two means of the normal distribution ($CI_{\mu_X-\mu_Y}$). The coverage probabilities and average lengths of the proposed confidence intervals were evaluated through Monte Carlo simulations.

For the single mean with unknown CV, the results are similar to the paper by Sodanin et al. [16] in terms of coverage probability and average length for all cases. The coverage probabilities of $CI_{GCI.\theta}$ were satisfactorily stable around 0.95. Therefore, $CI_{GCI.\theta}$ was preferred for the single mean of the normal distribution with unknown CV. $CI_{LS.\theta}$ and $CI_{LS.\theta^*}$ have the coverage probabilities under the nominal confidence level of 0.95 and close to 1.00 when σ increases. Therefore, the LS approach is not recommended to construct the confidence interval for the mean with unknown CV. Furthermore, CI_{μ} is better than $CI_{GCI.\theta}$ in terms of the average length when the sample size is small, whereas $CI_{GCI.\theta}$ is better than CI_{μ} in terms of coverage probability when the sample size is large. However, the coverage probability of $CI_{GCI.\theta}$ is more stable than that of CI_{μ} in all sample size cases. Therefore, the GCI approach is recommended as an interval estimator for the mean with unknown CV.

For the difference of two means with unknown CVs, the coverage probabilities of $CI_{GCI.\delta}$ satisfy the nominal confidence level of 0.95 for all cases. Therefore, $CI_{GCI.\delta}$ was preferred for the difference of the means with unknown CVs. The LS and MOVER approaches are not recommended to construct the confidence interval for the difference of means with unknown CVs. Furthermore, $CI_{GCI.\delta}$ is better than $CI_{\mu_X-\mu_Y}$ in terms of the coverage probability. Therefore, the GCI approach can be used to estimate the confidence interval for the difference of means with unknown CVs.

Hence, it can be seen in this paper that the new estimator of Srivastava [12] is utilized and well established both in constructing the single mean confidence interval and the difference of means of normal distributions when the CVs are unknown.

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