ST559 Homework 1

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1 Q1

We are given that if $\theta = 1$ then $y \sim N(1, \sigma)$ and if $\theta = 2$, then $y \sim N(2, \sigma)$. Furthermore, the distribution of θ is:

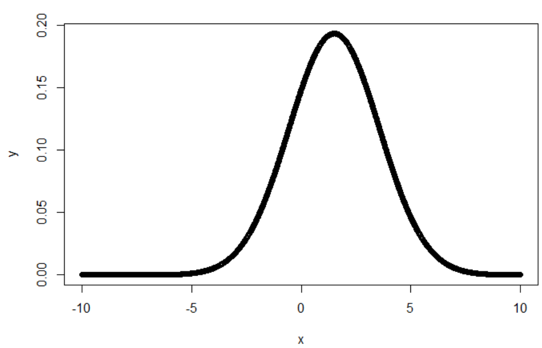
$$\theta = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ 2 & \text{w.p } \frac{1}{2} \end{cases}$$

1.1 part a.

For $\sigma = 2$ find the formula for the marginal pdf of y. Sketch it as well.

This distribution will essentially be an equal mixture distribution between $N(1, 4)$ and $N(2, 4)$.

$$\begin{aligned} f(y) &= f(y|\theta = 1)P(\theta = 1) + f(y|\theta = 2)P(\theta = 2) \\ &= \frac{1}{2}\left(\frac{1}{2\sqrt{2\pi}} exp\left(-\frac{(y-1)^2}{4}\right) + \frac{1}{2\sqrt{2\pi}} exp\left(-\frac{(y-2)^2}{4}\right)\right) \\ &= \frac{1}{4\sqrt{2\pi}} exp\left(-\frac{1}{4}\left(2y^2 - 6y + 5\right)\right) \end{aligned}$$



1.2 part b

Find $P(\theta = 1|y = 1)$, again assuming that $\sigma = 2$. Using Bayes Rule, we arrive at the following formula:

$$\begin{aligned} P(\theta = 1|y = 1) &= \frac{P(y = 1|\theta = 1)P(\theta = 1)}{P(y = 1)} \\ &= \frac{0}{P(y = 1|\theta = 1)P(\theta = 1) + P(y = 1|\theta = 2)P(\theta = 2)} \\ &= 0 \end{aligned}$$

Since this mixture distribution is continous, the probability y equals any single value is 0, regardless of whether or not we condition the value.

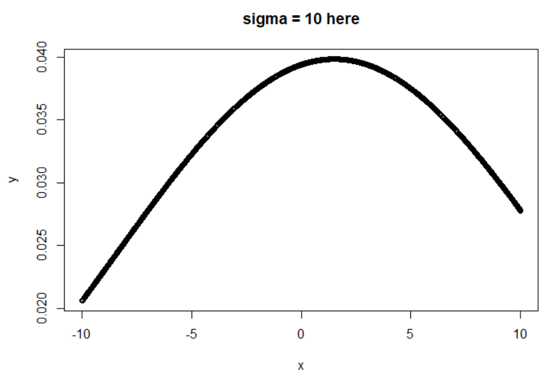
However, I think what this question is asking is to plug in the pdf value that one would get from the `dnorm` function in R.

$$\begin{aligned} P(\theta = 1|y = 1) &= \frac{P(y = 1|\theta = 1)P(\theta = 1)}{P(y = 1)} \\ &= \frac{.5\text{dnorm}(1, 1, 2)}{.5\text{dnorm}(1, 1, 2) + .5\text{dnorm}(1, 2, 2)} \\ &= .53 \end{aligned}$$

1.3 part c

Describe how the posterior density of θ changes in shapes as σ is increased or decreased.

After playing around with different values of σ, we see that as σ increases, the curve of the mixture distribution becomes less like a bell shape and more like a simple concave surface.



This tells us that as $\sigma \to \infty$, the posterior density becomes more like the prior and a single observation does little to change the probability that θ equals any specific value. As $\sigma \to 0$, the posterior density becomes very focused around 1.

2 Q2

Does the law of total expectation and total variance hold for vectors?

Expectation For the law of total expectation, **yes it holds**. If we are given a vectors $\vec{u}$ and $\vec{v}$, consider $E[\vec{u}] = E[E[\vec{y}|\vec{v}]]$. We can just take the expectations (either integrals or summations) componentwise in the vector and see that the law holds.

Variance For the law of total variance, we need to consider both the diagonal elements and the off-diagonal elements. For the diagonal elements which should be the variances of each individual component, which according to the univariate law of total variance is $var(u_i) = E[var(u_i|v)] + var(E[u_i|v])$.

For the off diagonal elements, this is slightly more complicated since we have to deal with covariances. Basically, we want to take two distinct elements u_i, u_j and insert those elements into the formula for the law of total variance. At the end, we should have $cov(u_i, u_j)$.

$$
\begin{aligned}
&E[cov(u_i, u_j|v)] + cov(E[u_i|v] + E[u_j|v]) \\
&= E[E[u_iu_j|v] - E[u_i|v]E[u_j|v]] + E[E[u_i|v]E[u_j|v]] - E[E[u_i|v]]E[E[u_j|v]] \\
&= E[u_iu_j] - E[u_i]E[u_j] \\
&= cov(u_i, u_j)
\end{aligned}
$$

So the law of total variance does hold for vectors!

3 Q3

Eye color is assigned by two alleles, X and x. Assume that the probabiliy of an allele being $x = p$. A person has blue eyes only if they have xx alleles. If a person is a heterozygote, they have brown eyes.

Heterozygotes, given child has brown eyes and parents have brown eyes Using Bayes rule, we want to split this conditional probability into a fraction. The numerator will be the probability that the child is heterozygous given the child and parents have brown eyes. The denominator will be the probability that the child will have brown eyes given that the parents have brown eyes.

It will be useful to use to have a bunch of conditional probability written out in terms of p.

$$
\begin{aligned}
P(Xx|\text{parents both xX})P(\text{parents both xX}) &= \frac{1}{2}(2p(1-p))^2 \\
P(Xx|\text{one parent is xX})P(\text{one parent is xX}) &= 2p(1-p)(1-p)^2 \\
P(Xx|\text{both parents XX})P(\text{both parents XX}) &= 0 \\
P(\text{brown eyes}|\text{both parents XX}) &= (1-p)^4 \\
P(\text{brown eyes}|\text{one parent Xx}) &= 2(1-p)^2(2p(1-p))2 \\
P(\text{brown eyes}|\text{both parents are Xx}) &= \frac{3}{4}(2(1-p)(p))^2
\end{aligned}
$$

Putting this all together, we get

$$
P(xX|\text{child has brown eyes, parents have brown eyes}) = \frac{2p(1-p) + 2p^2}{(1-p)^2 + 4p(1-p) + 3p^2} = \frac{2p}{1+2p}
$$

Judy's kiddos In this scenario, Judy is the brown eyed child of brown eyed parents and marries a heterozygote. They have n children who all have brown eyes. We want to find the posterior probability that Judy is a heterozygote and the probability that her first grandchild has blue eyes.

Posterior probability I will do a simpler case first. Assume that $n = 1$, then the probability that Judy is heterozygous given her child has brown eyes and husband is heterozygous can be found using Bayes' rule and the probability that we found in the above section.

$$
\frac{\frac{3}{4}\frac{2p}{2p+1}}{\frac{3}{4}\frac{2p}{2p+1} + \frac{1}{2p+1}}
$$

To extend this result to n children, using the assumption that the children are independent from each other, we arrive at the following:

$$
P(xX|\text{husband is xX, children has brown eyes}) = \frac{\frac{3}{4}^n\frac{2p}{2p+1}}{\frac{3}{4}^n\frac{2p}{2p+1} + \frac{1}{2p+1}}
$$

First grandchild has blue eyes Since we know that Judy's children all have brown eyes, Judy's children can either be heterozygotes or have XX alleles. However, Judy's grandchildren can **only** have have blue eyes if Judy's children are heterozygotes. Therefore, the first thing we need to calculate is the probability that Judy's child will be heterozygotes. By conditioning on whether or not Judy is a heterozygote herself, we get

$$\begin{aligned} P(\text{Judy's child is xX}) &= \frac{2}{3}\frac{\frac{3}{4}^n\frac{2p}{2p+1}}{\frac{3}{4}^n\frac{2p}{2p+1}+\frac{1}{2p+1}} + \frac{1}{2}\frac{\frac{1}{2p+1}}{\frac{3}{4}^n\frac{2p}{2p+1}+\frac{1}{2p+1}} \\ &= \left(\frac{3}{4}^n\frac{2p}{2p+1}+\frac{1}{2p+1}\right)^{-1}\left(\frac{3}{6}^n\frac{2p}{2p+1}+\frac{1}{4p+2}\right) \end{aligned}$$

Now assuming that Judy's child marries someone randomly, we can find the probability that Judy's grandchild has blue eyes. We need to condition on both Judy's child being heterozygous as well as whether or not Judy's child's spouse has xx or Xx alleles. This gives us the final answer.

$$\left(\frac{3}{4}^n\frac{2p}{2p+1}+\frac{1}{2p+1}\right)^{-1}\left(\frac{3}{6}^n\frac{2p}{2p+1}+\frac{1}{4p+2}\right)\left(\frac{1}{2}2p(1-p)+\frac{1}{4}p^2\right)$$

Super ugly lol.

4 Q6

We are given that approximately 1/125 of all births are fraternal twins and 1/300 births are identical twins. Elvis Presley had a twin brother. What is the probability that Elvis was an identical twin?

$$\begin{aligned} &P(\text{identical twin}|\text{twin brother}) = \\ &\frac{P(\text{twin brother}|\text{identical twin})P(\text{identical twin}}{P(\text{twin brother})|\text{identical twin})P(\text{identical twin})+P(\text{twin brother}|\text{fraternal twin})P(\text{fraternal twin})} \\ &= \frac{\frac{1}{2}\frac{1}{300}}{\frac{1}{2}\frac{1}{300}+\frac{1}{4}\frac{1}{125}} \\ &= \frac{5}{11} \end{aligned}$$

5 Q8

The probability of event E is **subjective** if two rational persons A and B if two rational persons can assign unequal probabilities to E, $P_A(E)$, and $P_B(E)$.

These probabilities can be interpreted as **conditional** $P_A(E) = P(E|I_A)$ and $P_B(E) = P(E|I_B)$ where I_A and I_B represent the knowledge available to A and B respectively.

Dice roll "The probability that a 6 appears when a fair dice is rolled, where A observes the outcome of the die roll and B does not."

This probability is not subjective since the probability of 6 on a fair dice is the same regardless of previous observed rolls. This posterior probability will be conditional though, since the person who observed the dice roll might have their beliefs influenced by the roll.

For example, if A observes that the dice roll is a 6, they might assume that the dice is loaded somehow and will roll a six more often than a fair die. This would contrast with B who does not observe anything and has no "reason" to believe the die is anything other than fair. If we express the probability of rolling a 6 using Bayes rule, we could get a formula similar to the following which instead focuses on the probability of having a fair die.

$$P(\text{Fair die}|\text{Roll a 6}) = \frac{P(\text{Roll a 6}|\text{Fair die})P(\text{Fair die})}{P(\text{Roll a 6})}$$

We cannot calculate this probability directly since we don't have any data, but we could think about what we can assume, for example reasonable priors like a fair die with equiprobable outcomes.

World Cup "The probability that Brazel wins the next World Cup where A is ignorant of soccer and B is a knowledgeable sports fan."

This probability is certainly conditional since the individual's estimated probabilities of Brazil winning is not independent on what they know about soccer. If the sports fan knows that Brazil is historically a world championship contender, they might assign a higher probability to Brazil winning than other individuals. This probability is also then subjective, since different individuals assign different probabilities to these same events. This is evident in modern day sports reporting where knowledgable people arrive at different predicted values and outcomes for future events.