Assignment 4

ST623

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Question 1

Normal Distribution

The pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ can be rewritten as $f(x) = e^{\frac{\mu x - \frac{\mu^2}{2}}{\sigma^2} - \left(\frac{x^2}{2\sigma^2} - \frac{1}{2} ln(2\pi\sigma^2)\right)}$.

This gives us:

- $\theta = \mu$
- $b(\theta) = \frac{\mu^2}{2}$
- $a(\phi) = \sigma^2$
- $c(x, \phi) = \frac{x^2}{2\sigma^2} - \frac{1}{2} ln(2\pi\sigma^2)$

with the mean, variance, and canonical link functions being:

- $\mu = b'(\theta) = \mu$
- $\sigma^2 = b''(\theta)a(\phi) = 1(\sigma^2) = \sigma^2$
- $g(\mu) = g(b'(\theta)) = g(\theta) = \theta = \mu$

Binomial Distribution

The pmf $f(x) = \binom{n}{x} p^x (1-p)^n$ can be rewritten $f(x) = e^{ln(\binom{n}{x}) + xln\left(\frac{p}{1-p}\right) + nln(1-p)}$.

This gives us:

- $\theta = ln\left(\frac{p}{1-p}\right)$ with $p = \frac{e^\theta}{1+e^\theta}$
- $b(\theta) = nln(1 + e^\theta)$
- The dispersion parameter ϕ doesn't matter

This gives us the mean, variance, and canonical link functions:

- $b'(\theta) = \frac{ne^\theta}{1+e^\theta} = np = \mu$
- $b''(\theta)a(\phi) = n\frac{e^\theta}{1+e^\theta}\left(\frac{1}{1+e^\theta}\right) = np(1-p)$
- $g(b'(\theta)) = g\left(\frac{ne^\theta}{1+e^\theta}\right) = ln\left(\frac{p}{1-p}\right)$ gives us the canonical link function $g(\mu) = ln\left(\frac{\mu}{n-\mu}\right)$

Poisson Distribution

The pmf $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ can be rewritten as $f(x) = e^{xln(\lambda) - \lambda - ln(x!)}$

This gives us:

- $\theta = ln(\lambda)$ and $\lambda = e^{\theta}$
- Dispersion parameter ϕ doesn't matter
- $b(\theta) = e^{\theta}$
- $c(x, \phi) = ln(x!)$

This gives us the following functions

- $b'(\theta) = e^{\theta} = \lambda$
- $b''(\theta)a(\phi) = \lambda$
- $g(b'(\theta)) = g(e^{\theta}) = \theta$ gives us $g(\mu) = ln(\mu)$

Exponential Distribution

- The pdf $f(x) = \frac{1}{\lambda}e^{-\frac{x}{\lambda}}$ can be rewritten as $e^{-\frac{x}{\lambda} - ln(\lambda)}$

This gives us

- $\theta = -\frac{1}{\lambda}$ and $\lambda = -\frac{1}{\theta}$
- $b(\theta) = -ln(\lambda) = -ln(-\frac{1}{\theta})$
- The dispersion parameter ϕ can be anything

Giving us the following mean, variance, and canonical link functions:

- $b'(\theta) = -\theta(\theta^{-2}) = \lambda$
- $b''(\theta)a(\phi) = \lambda^2$
- $g(b'(\theta)) = g(-\frac{1}{\theta}) = \theta$ means $g(\mu) = -\frac{1}{\mu}$

Inverse Gamma Distribution

- The pdf $f(x) = \sqrt{\frac{\lambda}{2\pi y^3}}exp\left(-\frac{\lambda(y-\mu)^2}{2\mu^2 y}\right)$ can be rewritten in:

$$f(x) = exp\left(\frac{x\frac{1}{2\mu^2} - \frac{1}{\mu}}{-\frac{1}{\lambda}} - \frac{\lambda}{2x} + \frac{1}{2}ln(\lambda) - \frac{1}{2}ln(2\pi x^3)\right)$$

This gives us

- $\theta = \frac{1}{2\mu^2}$ which gives us $\mu = \sqrt{\frac{1}{2\theta}}$
- $b(\theta) = \frac{1}{\mu} = \sqrt{2\theta}$
- The dispersion parameter $\phi = \lambda$ and $a(\phi) = -\frac{1}{\lambda}$

and the mean, variance, and canonical link functions

- $b'(\theta) = \frac{1}{\sqrt{2\theta}} = \mu$
- $b''(\theta)a(\phi) = -(2\theta)^{-3/2}(-\frac{1}{\lambda}) = \frac{2\theta^{-3/2}}{\lambda} = \frac{\mu^3}{\lambda}$
- $g(b'(\theta)) = g(\frac{1}{\sqrt{2\theta}}) = \theta$ means $g(\mu) = \frac{1}{2\mu^2}$

Negative Binomial Distribution

The pmf $f(x) = \binom{x+r-1}{x}(1-p)^x p^r$ can be rewritten as $f(x) = exp(yln(1-p) - (-rln(p)) + ln(\binom{x+r-1}{r}))$

This gives us

- $\theta = ln(1-p)$ which means $p = 1 - e^{\theta}$
- $b(\theta) = -rln(1-e^{\theta})$
- The dispersion parameter ϕ doesn't matter

and the mean, variance, and canonical link functions are

- $b'(\theta) = \frac{re^{\theta}}{1-e^{\theta}} = \frac{r(1-p)}{p}$
- $b''(\theta)a(\phi) = \frac{r(e^{\theta}-e^{2\theta})+re^{2\theta}}{(1-e^{\theta})^2} = \frac{re^{\theta}}{(1-e^{\theta})^2} = \frac{r(1-p)}{p^2}$
- $g(b'(\theta)) = g\left(\frac{re^{\theta}}{1-e^{\theta}}\right) = \theta$ means $g(\mu) = ln\left(\frac{\mu}{r+\mu}\right)$

Question 2

Consider the exponential distribution from 1d. We know that the canonical link function $g()$ is $g(\mu) = -\frac{1}{\mu}$.

If we let $z_i = x_i^T\beta + (y_i - \mu_i)g'(\mu_i)$, the score function for the MLE is

$$S(\beta) = \sum w_i(z_i - x_i^T\beta)x_i = 0$$

This can be rewritten in matrix form as $X^TWz = X^TWX\beta$

The matrix of weights will be a diagonal matrix where the nonzero entries $w_i = \frac{1}{a_i(\phi)b''(\theta_i)(g'(\mu_i))^2}$. Since this is the exponential distribution, we know that $a_i(\phi) = 1$, $b''(\theta_i) = \lambda^2$, and $[g'(\mu_i)]^2 = \left(\frac{1}{\mu_i^2}\right)^2$